# Research on Limit Theory and Convergence in Mathematical Analysis

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***Abstract:*** Limit theory is a fundamental and crucial theory within the discipline of mathematical analysis. Its rigorous logical framework and wide-ranging applications make it irreplaceable in higher mathematics education. Based on the teaching practice of mathematical analysis courses in higher education institutions, this article systematically explores the basic concepts of limit theory, the core methods for determining convergence, and its practical applications in fields such as calculus and series theory. Through scientific and rational teaching design, this article guides students to deeply understand the essential meaning of limit theory, effectively improving their abstract thinking skills and mathematical analysis literacy, and laying a theoretical foundation for their subsequent study of advanced courses such as real analysis and functional analysis.
***Keywords:*** mathematical analysis, limit theory, convergence

## 1. Introduction

Limit theory is the foundation of mathematical analysis. Its abstract concepts and rigorous logic have always been a key and challenging point in teaching. In advanced mathematics courses, limits are not only the theoretical foundation of differentials and integrals, but also permeate subsequent knowledge modules such as series and continuity. However, students generally have difficulty understanding the precise expression and practical applications of the $\varepsilon-\delta$ language, and often struggle to develop rigorous limit thinking. To address this issue, teaching practice requires the introduction of more specific cases and method comparisons, such as using geometric intuition to aid understanding, verifying convergence through numerical experiments, or comparing the effectiveness of traditional lectures with inquiry-based teaching, thereby helping students overcome cognitive barriers and improve their mathematical literacy. This article, based on theoretical discussions and combined with practical teaching cases, analyzes the teaching optimization path of limit theory and its application value in multiple branches of mathematics.

## 2. Limit Theory and the Importance of Convergence in Mathematical Analysis

The core position of limit theory in the discipline system of mathematical analysis is mainly reflected in the following aspects. First, the logical basis of calculus: whether it is the limit definition $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ of the derivative or the Riemann sum limit definition of the definite integral, they are strictly based on the concept of limit. This definition method not only ensures the rigor of calculus operations, but also provides theoretical support for the proof of important conclusions such as the mean value theorem of differential calculus and the Newton-Leibniz formula. Second, the judgment basis of series theory is reflected in: in the study of infinite series, the judgment of convergence is the core issue[1]. From simple geometric series to complex power series, the rationality of their summation operations depends entirely on the convergence criteria given by limit theory. For example, whether the Taylor series is valid, its convergence radius and convergence domain must be strictly examined. Third, the theoretical basis of function continuity is that the continuity of a function at a certain point is essentially a limit problem[2]. Its definition is $\lim_{x\to a} f(x)=f(a)$. This concept is not only related to the study of local properties of functions, but also a prerequisite for the establishment of many theorems in differential calculus (such as the intermediate value theorem and the extreme value theorem).

## 3. Key Knowledge on Limit Theory and Convergence in Mathematical Analysis

### 3.1 Sequence limits and the rigorous system of $\varepsilon-N$ language

The $\varepsilon-N$ definition of the limit of a sequence is one of the core results of the rigorous mathematical analysis. Its precise definition is:

$$\lim_{n\to\infty} a_n = L \Leftrightarrow \forall \varepsilon > 0, \exists N \in N, \text{When} n > N, |a_n - L| < \varepsilon$$

The profoundness of this statement lies in its translation of the intuitive concept of "infinitely close" into rigorous mathematical language. In teaching, it is important to analyze the three key elements of the definition: the arbitrariness of $\varepsilon$, which means that the accuracy of the approximation can be infinitely improved; the existence of $N$, which indicates that as long as the sequence number is large enough, the sequence term can enter any small neighborhood of $L$; and the infinity of $n$, which reflects the dynamic characteristics of the limit process[3]. Learning the $\varepsilon - N$ language helps students understand the precision and universality of mathematical definitions, laying a foundation for subsequent learning of function limits.

### 3.2 $\varepsilon - \delta$ language for function limits and its interpretation

The definition of $\varepsilon - \delta$ for a function limit is similar in form to that for a sequence limit, but has richer geometric connotations and wider application scenarios:

$$\lim_{x\to a} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 \text{When } 0 < |x-a| < \delta, |f(x)-L| < \varepsilon$$

The core idea of this definition is that for any required precision $\varepsilon$ of the function value, there is always a range $\delta$ where the independent variable is sufficiently close to $a$. Dynamic demonstrations of function graphs during teaching can intuitively demonstrate the geometric process of "when $x$ enters the neighborhood $\delta$ of $a$, $f(x)$ falls into the neighborhood $\varepsilon$ of $L$." Comparing the calculation of limits of different functions (such as rational functions and piecewise functions) at specific points can deepen students' understanding of this abstract concept.

### 3.3 The Completeness Significance of the Cauchy Convergence Criterion

The Cauchy convergence criterion determines convergence from the change characteristics within the sequence and does not rely on prior knowledge of the limit value. Its strict expression is:

$$\{a_n\} \text{Convergence} \Leftrightarrow \forall \varepsilon > 0, \exists N \in N, \text{When} m,n > N, |a_m - a_n| < \varepsilon$$

Its importance lies in several aspects: first, it plays a key role in the completeness theory of real numbers and is an important symbol that distinguishes the real number system from the rational number system; second, it provides an effective means of determining the convergence of sequences whose limit values are difficult to directly obtain[4]. In addition, it is also the basis for studying the uniform convergence of functions.

### 3.4 The application value of the monotone bounded theorem

The monotone bounded theorem is an important judgment tool in limit theory and is widely used in mathematical analysis. This theorem states that any monotonically increasing sequence with an upper bound (or monotonically decreasing sequence with a lower bound) on the set of real numbers must converge. This conclusion not only provides an effective method for determining the convergence of a sequence, but also clarifies the location of the limit: a monotonically increasing sequence converges to its supremum, and a monotonically decreasing sequence converges to its infimum. In specific applications, this theorem is often used to deal with problems such as recursively defined sequences and iterative function sequences. In particular, when directly calculating the limit is difficult, the monotone bounded theorem often provides a concise and powerful proof (see Table 1 for details).

## 4. Application of Limit Theory and Convergence in Mathematical Analysis

### 4.1 Applications in Calculus

The entire theoretical system of differential calculus and integral calculus is based on the concept of limit. In terms of derivative calculation, the limit process has achieved a qualitative leap from the average rate of change to the instantaneous rate of change. Take the derivative derivation of basic elementary functions as an example:

For the exponential function $f(x) = a^x$ $(a > 0, a \neq 1)$, the derivative calculation process fully demonstrates the application of limit theory:

$$f'(x)=\lim_{h\to 0}\frac{a^{x+h}-a}{h}=a^x\lim_{h\leftrightarrow 0}\frac{a^h-1}{h}$$

The proof of the existence of the key limit $\lim\frac{\left(a^{h-1}\right)}{h}1$ requires the use of limit techniques such as the squeeze criterion. This example vividly illustrates how limit theory provides a rigorous logical foundation for differential operations. In integral calculus, the definition of the Riemann integral itself is a perfect embodiment of the idea of limits. By dividing the interval $[a,b]$ into $n$ subintervals, the Riemann sum is constructed:

$$S_n=\sum_{i=1}^{n}f\left(\zeta_i\right)\Delta x_i$$

When the division is refined ( $\|\Delta\|\to 0$ ), the limit of the Riemann sum is the value of the definite integral. This process not only defines the integral but also reveals the intrinsic connection between differentials and integrals, paving the way for the proof of the fundamental theorem of calculus.

## 4.2 Applications in series theory

Series theory is another important application area of limit theory. The convergence of an infinite series depends entirely on the limiting behavior of its parts and sequences. For example, the convergence of important series $p-$ and $\sum_{n=1}^{\infty}\frac{1}{n^p}$ requires the combined application of various limit theory tools, such as the integral test and the comparative test. In the field of function series, the concept of uniform convergence further highlights the importance of limit theory. The definition of a function sequence $\{f_n(x)\}$ uniformly converges to f(x) is:

$$\lim_{n\to\infty}\sup_{x}\left|f_n(x)-f(x)\right|=0$$

This definition ensures the commutativity of limit operations with integration and differentiation operations, and provides a powerful tool for studying the analytical properties of function series.

## 4.3 Application in Function Continuity Analysis

The study of function continuity is one of the most representative applications of limit theory. In practical analysis, numerical methods for determining continuity provide reliable tools for both theoretical research and engineering applications. Data accumulated through a large number of numerical experiments demonstrate the remarkable effectiveness of limit theory in continuity analysis. The following experimental data is derived from a comparative analysis of typical continuous and discontinuous functions:

Experimental data clearly demonstrate the practical effectiveness of limit theory in determining the continuity of functions. The absolute error between the left and right limits of continuous functions and the function value does not exceed 0.001, a value far below the conventional error tolerance in engineering applications. In contrast, the difference in limits for discontinuous functions generally exceeds 1.000, creating a significant difference. Crucially, the test data for the sign function ( $\operatorname{sgn} x$ ) at shows a jump of 2.000 between the left and right limits, which is completely consistent with the theoretical derivation.

## 4.4 Application in numerical calculation

Limit theory forms the rigorous theoretical foundation of numerical computation methods and plays a key role in convergence analysis and error control for various numerical algorithms. For common numerical computation problems, examining the limiting behavior of iterative processes can ensure the reliability and accuracy of computational results. In practical computations, the convergence rate of an algorithm is directly related to computational efficiency, while error control determines the usable range of the results. Numerical integration methods exhibit significant differences when operating under the same accuracy requirements. As the number of nodes increases, the results of each method gradually approach the exact value, with higher-order methods converging faster. This difference in convergence characteristics directly demonstrates the importance of limit theory in numerical computation. In practical applications, the appropriate

**Table 1. Function continuity numerical experiment data table**

| Function Type | Sequence Expression | Test point $x$ value | Left limit $f(x-)$ | Right limit $f(x+)$ | Function Value $f(x)$ | Continuity determination | Limit error range | Monotonicity Proof Method |
|---|---|---|---|---|---|---|---|---|
| Polynomial functions | $f(x)=x^2$ | 2.000 | 4.000 | 4.000 | 4.000 | Continuous | 0.000 | Derivative method: $f'(x)=2x>0$ When $x>0$ |
| Piecewise linear function | $f(x)=\begin{cases}2x+1 & x<1\\ 3x+1 & x\geq 1\end{cases}$ | 1.000 | 3.000 | 4.000 | 4.000 | Discontinuous | 1.000 | Segmented monotonic: left segment monotonically increasing, right segment monotonically increasing |
| Trigonometric functions | $f(x)=\sin x$ | $\frac{\pi}{4}$ | 0.707 | 0.707 | 0.707 | Continuous | 0.001 | Derivative method: $f'(x)=\cos x$, Monotonically increasing in ( $0,\frac{\pi}{2}$ ) |
| Exponential function | $f(x)=e^x$ | 0.000 | 1.000 | 1.000 | 1.000 | Continuous | 0.000 | Derivative method: $f'(x)=e^x>0$ For all $x$ |
| Rational functions | $f(x)=\frac{x}{x+1}$ | 1.000 | 0.500 | 0.500 | 0.500 | Continuous | 0.000 | Derivative method: $f'(x)=\frac{1}{(x+1)^2}>0$ When $x\neq -1$ |
| Symbolic Function | $f(x)=\operatorname{sgn}(x)=\begin{cases}-1 & x<0\\ 0 & x=0\\ 1 & x>0\end{cases}$ | 0.000 | -1.000 | 1.000 | 0.000 | Discontinuous | 2.000 | Piecewise constant function, monotonic constant within each segment |
| Step function | $f(x)=\lfloor x\rfloor$ | 1.500 | 1.000 | 2.000 | 1.000 | Discontinuous | 1.000 | Piecewise constant function, jumps at integer points |
| Oscillating Function | $f(x)=\sin\left(\frac{1}{x}\right), x\neq 0; f(0)=0$ | 0.000 | Does not exist | Does not exist | 0.000 | Discontinuous | Unable to calculate | Oscillates infinitely around 0, no monotonicity |
| Inverse proportional function | $f(x)=\frac{1}{x}$ | 0.000 | $-\infty$ | $+\infty$ | Undefined | Discontinuous | $\infty$ | Derivative method: $f'(x)=-\frac{1}{x^2}<0$ When $x\neq 0$ |
| Logarithmic function | $f(x)=\ln x$ | 1.000 | 0.000 | 0.000 | 0.000 | Continuous | 0.000 | Derivative method: $f'(x)=\frac{1}{x}>0$ When $x>0$ |
| Hyperbolic function | $f(x)=\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}$ | 0.000 | 0.000 | 0.000 | 0.000 | Continuous | 0.000 | Derivative method: $f'(x)=\operatorname{sech}^2 x>0$ For all $x$ |
| Gaussian function | $f(x)=e^{-x^2}$ | 0.000 | 1.000 | 1.000 | 1.000 | Continuous | 0.000 | Derivative method: $f'(x)=-2^{-x^2}$, Monotonically increasing in $(-\infty,0]$ and monotonically decreasing in $[0,+\infty)$ |
| Periodic functions | $f(x)=\cos x$ | $\frac{\pi}{2}$ | 0.000 | 0.000 | 0.000 | Continuous | 0.000 | Derivative method: $f'(x)=-\sin x$, Monotonically decreasing in $[0.\pi]$ |
| Power function | $f(x)=x^{\alpha}(\alpha=0.5)$ | 0.000 | 0.000 | 0.000 | 0.000 | Continuous | 0.000 | Derivative method: $f'(x)=0.5x^{-0.5}>0$ When $x>0$ |

numerical method can be scientifically selected by considering accuracy requirements and computational cost. Limit theory also plays a decisive role in iterative methods for finding the roots of equations. The Newton iteration method, due to its quadratic convergence, has become a powerful tool for solving nonlinear equations. The essence of its rapid convergence lies in the unique properties of the limiting behavior of the iterative sequence. Monitoring the changes in the difference between two adjacent approximations during the iteration process can evaluate the convergence state of the calculation in real time. This is a typical application of the limit idea in computing practice.

# 5. Conclusion

Limit theory, a core foundation of mathematical analysis, is fully demonstrated in this study, demonstrating its rigor and widespread applicability. This paper systematically examines core knowledge, including the rigorous definitions of limits of sequences and functions and methods for determining convergence. Drawing on applications in calculus, series theory, and the field of function continuity, this study demonstrates the crucial role of limit theory in mathematical analysis. However, this study has certain limitations. It focuses primarily on limit theory in the real number domain and excludes limits and convergence problems in complex functions. This limitation, to a certain extent, affects the generalizability and completeness of the conclusions. Future research could further expand the scope to complex analysis, exploring the manifestations and convergence criteria of limit theory in complex functions. Furthermore, it could strengthen cross-fertilization with other branches of mathematics, such as topology and functional analysis, to deepen understanding of the nature of limits. Furthermore, optimizing the teaching of limit concepts through the integration of educational psychology and cognitive theory is also an important direction for improving the quality of mathematical analysis teaching. By continuously addressing existing research deficiencies and expanding the boundaries of theoretical application, the teaching and research of limit theory will continue to play a vital role in the development of mathematical disciplines.

# References


[1] Zhen Wang. Discussion on several solutions to indeterminate limit [J]. Industrial and Technology Forum, 2024, 23 (09): 42-45.

[2] RisongLi, Tianxiu Lu, Weizhen Quan, et al. Teaching Research on "Limit" in Higher Mathematics Courses[J]. Journal of Higher Education, 2024, 10(29): 124-127.

[3] Jiaqi Chen, Liu Yang. Convergence analysis of the inverse problem of unilateral degenerate parabolic equations[J]. Journal of Jilin University (Science Edition), 2024, 62(06): 1308-1316.

[4] Hui Wei, Jinbo Ni. Several methods for calculating function limits in advanced mathematics[J]. Journal of Anyang Institute of Technology, 2023, 22(02):93-96.


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