Why the Sarrus Rule Stops at Order Three: A Mathematical Explanation and Its Teaching Implications

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Abstract: The Sarrus rule (or diagonal rule) is how most students first learn to calculate 2×2 and 3×3 determinants. It's simple, visual, and widely taught. Yet many assume it extends to 4×4 or larger determinants — and run into trouble. This paper explains why the rule necessarily stops at orders $n = 3$. The explanation draws on three perspectives. First, a combinatorial count: for $n \geq 4$, the $n!$ terms of the determinant cannot be captured by the $2n$ terms of any diagonal-type rule. Second, permutation groups: showing that the Sarrus rule captures only two specific subsets of the symmetric group S_n — a complete coverage for S_3 but a severe underrepresentation for $n \geq 4$. Third, Laplace expansion: the compact form of the rule for $n = 3$ is a special case of the cofactor expansion that has no analogue in higher dimensions. These findings suggest a natural teaching sequence. Students practice the rule on low-order examples, then try it on a 4×4 matrix where it fails. That failure opens the door to the formal definition and, finally, to general methods. This approach transforms a common student misconception into a foundation for understanding the structure of determinants.
***Keywords*:** Sarrus rule; determinant; mathematics education; linear algebra pedagogy

1. Introduction

1.1 Research Background and Problem Statement

Students first meet the Sarrus rule when learning to compute 2×2 and 3×3 determinants. With a 2×2 matrix, the calculation is straightforward cross multiplication. With a 3×3, the usual trick is to extend the matrix by copying the first two columns, then add products along the six diagonals.This method lets students compute determinants before they encounter the formal definition.

Yet a persistent difficulty arises when students move beyond order three. Many, struck by the elegance of the rule, naturally attempt to extend it to 4×4 or larger determinants. The result is almost invariably error. This recurring mistake points to two underlying issues. First, students' understanding of the determinant remains procedural, tied to a memorized pattern rather than to its mathematical substance. Second, standard instruction rarely addresses the question of why the Sarrus rule works where it does — and, just as importantly, why it fails where it does. The boundaries of the rule's applicability are seldom made explicit, leaving students to infer them incorrectly.

From a mathematical standpoint, the limitations of Sarrus rule are rooted in the definition of the determinant itself. Lorenz et al. [1] provide a rigorous group-theoretic proof: an n-order determinant contains of $n!$ terms, while Sarrus rule can only cover $2n$ terms. When $n = 2$, $2!=2$ and $2n = 4$ (the actually valid terms are 2); when $n = 3$, $3!=6$ and $2n = 6$; when $n \geq 4$, $n!$ grows at a factorial level ($4!=24$, $5!=120$), which far exceeds the linear growth of $2n(8,\ 10)$. Thus, the rule will inevitably fail due to the omission of a large number of terms. In other words, Sarrus rule is essentially a "coincidence" when $n = 3$ — it exactly enumerates all permutations of S, but this pattern cannot be replicated when $n \geq 4$.

Despite these limits, efforts to generalize the Sarrus rule to higher orders have persisted. Several authors proposed methods for 4×4 or 5×5 determinants, often by extending the matrix or adapting cofactor expansions [2–6]. None succeeded in preserving the simplicity of the original rule, and most still did not cover all the terms the determinant requires — a pattern that fits the theoretical result of Lorenz.

Teaching studies, meanwhile, have largely focused on how to present the definition of a determinant [7] or on difficulties with expansion theorems [8]. The specific question of why the Sarrus rule stops at order three — and how to teach this — has drawn little attention. Although mathematicians have known about the rule's limitations for a long time (Arschon's 1935 paper [9] is an early example), but that understanding has not made its way into classroom guidance. The mathematics is clear; the pedagogy is not.

The present paper aims to bridge this gap. It offers a mathematically rigorous account of why the Sarrus rule is necessarily confined to orders two and three, drawing on combinatorial counting, permutation group theory, and Laplace expansion. It then translates this analysis into a concrete pedagogical sequence, designed to help students move beyond a procedural grasp of the rule toward a deeper understanding of determinants as mathematical objects.

1.2 Research significance

Clarifying why the Sarrus rule applies only for 2×2 and 3×3 determinants does more than answer a narrow question. It reveals a deeper connection between intuitive low-dimensional methods and the abstract theories required in higher dimensions.

At the theoretical level, asking why the rule stops at order three forces a closer look at what a determinant really is. The three approaches here — combinatorial term counts, analyzing permutation groups, and using Laplace expansion — each help clarify the definition of a determinant as a signed sum over permutations. Understanding why a rule fails, in that sense, is also understanding why it works.

At the teaching level, a clear answer feeds directly into classroom practice. Students who master the diagonal method for 2×2 and 3×3 cases often assume it extends to higher orders. Instead of correcting that mistake after it happens, one can start with the question: "why can't we do the same for a 4×4?" Asking students to try — and watching it fail — creates a productive surprise. That moment of cognitive conflict opens the door to introducing the formal definition of an nth-order determinant. Letting students discover the limits of a familiar tool can be more effective than simply presenting abstract definitions from the outset.

1.3 Structure of the Paper

The remainder of this paper is organized as follows. Section 2 describes the Sarrus rule for second- and third-order determinants. Section 3 examines why the rule cannot be extended to higher orders — first through a combinatorial count, then through the lens of permutation groups, and finally by way of Laplace expansion. Section 4 uses these mathematical insights to propose a teaching sequence. Section 5 summarizes the main arguments.

2. The Sarrus Rule in Low Orders

For a 2×2 determinant $D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$,the Sarrus rule reducesto the familiar cross-multiplication: $D = a_{11}a_{22} - a_{12}a_{21}$, This expression consists of exactly $2! = 2$ terms, corresponding to the two permutations of $\{1,2\}$ — the identity $(1,2)$ with positive sign and the transposition $(2,1)$ with negative sign.

For a 3×3 determinant $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, students often learn the Sarrus rule through a visual trick: copy the first two columns to the right of the matrix, then sum products along the six diagonals. The calculation amounts to $D = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$. That is, the six terms in the expansion are precisely the six signed terms called for by the formal definition: three even permutations (the main diagonal products) and three odd ones (the anti-diagonal products).

In both cases, the Sarrus rule provides an enumeration of all terms required by the determinant definition. The key observation, however, is that this perfect match occurs only for $n = 2$ and $n = 3$.

3. Why Not Higher?

3.1 A Combinatorial Explanation: The Gap Between $n!$ and $2n$

For an $n \times n$ matrix $A = (a_{ij})$, the Leibniz formula defines the determinant as $\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$, where S_n is the symmetric group on $\{1, \cdots, n\}$. The sum consists of $n!$ terms, one for each permutation. A naive extension of the Sarrus rule to higher orders would generate only $2n$ products — n along the main diagonal and n along the anti-diagonal. Table 1 compares $n!$ with $2n$.

Table 1. Term counts: $n!$ vs. $2n$

n	n!	2n	Missing terms	Missing percentage
2	2	4	0 (effective)	0%
3	6	6	0	0%
4	24	8	16	66.7%
5	120	10	110	91.7%
6	720	12	708	98.3%

For $n = 2$, the four products collapse to two distinct ones, matching $2!$. For $n = 3$, $2n = 6$ coincides with $3!$. For $n \geq 4$, however, $n!$ grows factorially while $2n$ grows linearly, leaving most terms uncovered.

3.2 A Permutation Group Perspective: Why Only Two Types of Permutations?

The Leibniz formula expresses the determinant as a sum over the entire symmetric group S_n. The Sarrus rule, when viewed through this lens, corresponds to selecting only two specific subsets of S_n:

(1) The n terms along the main diagonals correspond to the cyclic permutations $(1,2,\cdots,n)$, $(2,3,\cdots,n,1)$, $\cdots$, $(n,1,\cdots,n-1)$.

(2) The n terms along the anti-diagonals correspond to the permutations obtained by reversing the order.

For $n = 3$, these two subsets together exhaust S_3: the three even permutations (the identity and two 3-cycles) appear on the main diagonal, and the three odd permutations (the transpositions) on the anti-diagonal.

For $n = 4$, S_4 has 24 permutations with diverse cycle structures: one identity, six transpositions, eight 3-cycles, six 4-cycles, and three double transpositions. Two diagonal directions yield at most eight permutations, no fixed set of diagonal lines can capture the full variety

3.3 Connection to Laplace Expansion

The Laplace expansion (cofactor expansion) provides a recursive way to compute determinants. Expanding along the first row: $\det(A) = \sum_{i=1}^{n} a_{1j} C_{1j}$, where $C_{1j} = (-1)^{1+j} M_{1j}$ is the cofactor of a_{1j}, and M_{1j} is the corresponding minor.

For $n = 3$, expanding along the first row gives:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Expanding each 2×2 minor and collecting terms yields the familiar six-term Sarrus expression. In other words, the Sarrus rule for $n = 3$ is the Laplace expansion condensed into a compact, visually simple form.

For $n \geq 4$, the expansion branches into n minors of order $n-1$, each requiring further expansion. The total number of terms reaches $n!$, with no way to collapse them into a simple diagonal pattern. The signs of the cofactors add another layer of complexity, one that a fixed set of diagonal lines cannot capture. Thus, the simplification that occurs at $n = 3$ does not extend to higher dimensions.

4. Pedagogical Implications

The analysis in Section 3 suggests a way to address the teaching problem raised at the outset. The four-stage sequence that follows is designed around that idea: use cognitive conflict to help students see why the Sarrus rule has limits, then guide them toward the general definition.

4.1 Stage 1: Consolidation — Mastering the Sarrus Rule for n = 2 and n = 3

Students first practice the Sarrus rule on familiar ground. For a 2×2 determinant, they quickly see it is just cross multiplication. For a 3×3, they learn the visual trick of copying the first two columns to the right and summing along the six diagonals. Working through examples like $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ (which comes out to 0) gives them confidence in the method.

4.2 Stage 2: Cognitive Conflict — Attempting the Sarrus Rule for n = 4

The instructor then asks students to apply the same pattern to a 4×4 matrix. A good example is $D = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix}$. Most students will extend the Sarrus rule by copying the first three columns and summing along the four "main" and four "anti" diagonals. Their calculation will yield something like $D_3 = 9264 - 9888 = -624$. But the true determinant is 0 (the rows are arithmetic progressions, hence linearly dependent). The mismatch creates a genuine surprise: a trusted visual method has failed.

4.3 Stage 3: Explanation — Bridging to the Formal Definition

The conflict naturally motivates the question: why does the Sarrus rule work for $n = 2,3$ but not for $n = 4$? The instructor now introduces the Leibniz definition of the determinant: $\det(A) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\prod_{i=1}^{n} a_{i,\sigma(i)}$. Students can see that for $n = 2$ and $n = 3$ the numbers match $(2! = 2,\ 3!=6$, and the diagonal products give exactly those terms). For $n = 4$, the sum requires $4! = 24$ terms, but a naive diagonal rule supplies only eight. The remaining 16 terms — such as $a_{11}a_{23}a_{32}a_{44}$ —are omitted, and their signs are not accounted for. Linking this back to Section 3.2, the eight diagonal products correspond to only two kinds of permutations (cyclic and their reverses), while S_4 contains many other cycle structures that contribute nonzero terms.

4.4 Stage 4: Transition — General Methods for Higher Orders

Once students recognize the limitations of the Sarrus rule, they are ready to appreciate general determinant computation methods. The instructor introduces:

(1) Laplace expansion, which systematically generates all $n!$ terms and reduces to the Sarrus rule when $n = 3$.

(2) Row reduction, which computes determinants efficiently using elementary row operations.

A follow-up exercise can ask students to recompute 4×4 example by row reduction, confirming that the true value is 0 and reinforcing that general methods, while less visually appealing, are universally applicable.

This sequence — practice, surprise, explanation, transition — turns a common mistake into a learning opportunity. The deeper point is that low-dimensional intuition has its limits; abstraction becomes necessary not as a complication, but as a natural next step.

5. Conclusion

Why does the Sarrus rule only work for 2×2 and 3×3 determinants? The answer follows from three observations. Combinatorially, the rule supplies $2n$ terms while the determinant needs $n!$; the numbers match only for $n = 2$ and $n = 3$. Permutation groups show the rule picks out only cyclic permutations and their reverses — a complete set for S_3, but only a fraction for $n \geq 4$. And Laplace expansion reveals that the compact form for $n = 3$ is a special case that has no higher-order analogue.

The teaching sequence in Section 4 — practice, surprise, explanation, transition— uses this insight to turn a common mistake into a learning opportunity. The larger lesson is that low-dimensional intuition has limits, and abstraction becomes necessary when those limits are reached.

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References

[1] Lorenz, D, Wirths, K. J. (2018). Sarrus rules for matrix determinants and dihedral groups. The College Mathematics Journal, 49(5), 333-340.

[2] Mitrinović, D, Mihajlović, D, Ulčar, J. (1960). Zbornik matematičkih problema sa prilozima i numeričkim tablicama II. Beograd: Naučna Knjiga.

[3] Bankier, J. D. (1961). The diagrammatic expansion of determinants. The American Mathematical Monthly, 68(8), 788-790.

[4] Sobamowo, M. G. (2016). On the extension of Sarrus' rule to n×n (n>3) matrices: Development of new method for the computation of the determinant of 4×4 matrix. International Journal of Engineering Mathematics, 2016, 9382739.

[5] Salinas-Hernández, E, et al. (2021). Sarrus rule extension for 4×4 and 5×5 determinants. International Journal of Algebra, 15(4), 283-301.

[6] Zhuo Muxiang. (2020). Diagonal expansion method of fourth order determinant based on algebraic cofactor. Mathematical Learning and Research, (1), 10-11.

[7] Zeng Zhenbing, Huang Yong, Rao Yongsheng. (2020). Discuss determinant teaching from the angle of educational mathematics. Advanced Mathematical Studies, 23(4), 10-21.

[8] Yang Yuping, Song Keyan. (2021). The teaching difficulty and solution of determinant expansion theorem in linear algebra. Journal of Southwest Normal University (Natural Science Edition), 46(6), 190-194.

[9] Arschon, S. (1935). Verallgemeinerte Sarrussche Regel [Generalized Sarrus rule]. Matematicheskii Sbornik, 42(1), 121-128.