Research on Asian Option Pricing Based on Uncertain Volatility

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Abstract: In this paper, we study the impact of introducing uncertainty volatility into Asian options pricing, with emphasis on the use of Hamilton-Jacobi-Bellman (HJB) equation. The traditional Asian option pricing model usually assumes that volatility is known and constant, but in the actual market, volatility is often uncertain and volatile. This paper first reviews the pricing theory of Asian options, and then introduces the hypothesis of uncertain volatility. By constructing the HJB equation based on uncertainty volatility, a new pricing method is proposed and verified by numerical simulation. The results show that after the introduction of uncertain volatility, the price range of Asian options expands significantly, reflecting higher market uncertainty and risk.

***Keywords*:** Asian option, HJB equation, option pricing, uncertain volatility

1. Introduction

A weather derivative is a financial contract whose income depends on changes in the underlying meteorological index. There are also difficulties in pricing weather derivatives. First, the market for weather derivatives is incomplete, as the underlying indices (e.g., temperature, humidity, precipitation, wind, etc.) are not tradable, so there is no transferable price. Second, weather models do not follow geometric Brownian motion, so the methods in the existing Black-Scholes framework cannot be used directly for pricing weather derivatives[1].In addition, weather contracts are usually generated on some cumulative average or total run sum, which creates additional difficulties for practitioners in the weather derivatives market.

Temperature is the most widely used index in the weather derivatives market.Much of the literature [2-3] relates to temperature modeling and pricing of temperature derivatives.In these works, the mean regression Ornstein-Uhlenbeck process model proposed by Alaton [4] has been widely accepted. In [4], Alaton also developed two approximate formulas for heating degree-day (HDD) calls and placed them in cold areas such as Stockholm. However, it should be noted that these approximate formulas are of limited use for regions with other climatic conditions, as they are only derived under the prerequisite of cold weather.

Until recent years, some methods based on partial differential equations have gradually appeared in temperature derivative pricing. Harris [5] established and solved the partial differential equations of cumulative disk and temperature respectively in 2003,and developed a numerical scheme for the partial differential equation of the Ornstein-Uhlenbeck process with central difference to approximate the convective term, and compared this method with actuarial and consumption-based methods [6,7]. Broni-Mensah (2012) [8] derived the partial differential equation of weather selection by introducing a hedging tool H that is not completely related to temperature. Based on the incomplete hedging strategy, Tangang and Chang [9] proposed a weather-selective PIDE (partial integral differential equation) model controlled by mean reversion Brownian motion with jump diffusion, and proposed a semi-Lagrangian method for solving PIDE. Li [10] calculates the price of weather derivatives by solving the partial differential equation (PDE) of the Ornstein-Uhlenbeck process.

The HJB equation (Hamilton-Jacobi-Bellman equation) plays an important role in finance, especially in optimization and dynamic decision problems. It is the core equation of optimal control theory and is used to solve a class of problems called Dynamic Programming. By describing how the value of the optimal control strategy evolves over time, it is used to deal with portfolio optimization and derivatives pricing.

2. The basic concept of weather derivatives

Weather derivatives are typically structured as swaps, futures and options based on different underlying weather indices. In this paper we focus on the analysis of weather derivatives for HDD.

Given a weather station, set $x_i^{\max}$ and $x_i^{\min}$ to be the highest and lowest temperature measured in a day i , respectively. The DAT of the i day is defined as:

$$x_i = \frac{x_i^{max} + x_i^{min}}{2}$$

2.1 The introduction of the HDD

HDD is the number of times a day when the average temperature is below the baseline temperature.

$$HDD_i = \max\{x_{base} - x_i, 0\} = (x_{base} - x_i)^+,$$

where x_{base} is the base temperature. Let the contract period consist of n days.The HDD indexes $y_H(n)$ and $y_C(n)$ are

$$y_H(n) = \sum_{i=1}^{n} HDD_i,$$

In continuous form, the above quantities can be expressed as

$$y_H(t) = \int_0^t (x_{base} - x_t)^+ dt,$$

2.2 Weather option

For the HDD put option example,the payoff is

$$P(y_H(T), T) = tick \times (K - y_H(T))^+,$$

where $y_H(T)$ is the value of the HDD index at maturity, k is the strike level.

For a HDD call option, its payoff is given by $P(y_H(T), T) = tick \times (y_H(T) - K)^+$, $y_H(T)$, $tick$ and k are the same as the above.

2.3 HJB equation

The HJB equation is used to solve the problem of optimal allocation of a portfolio, especially when the returns and risks of a portfolio change over time. Investors want to maximize the expected return of the portfolio under a given level of risk, and HJB equation can be used to derive the optimal investment strategy.

3. Weather option's models

3.1 Temperature models

The basic framework for temperature modeling is based on the Ornstein-Uhlenbeck process

$$dx_t = dx_t^m + \alpha(x_t^m - x_t)dt + \sigma_t dW_t,$$

Where x_t^m is the long-term average of the process, α is the average return rate, σ_t is the volatility of the fluctuations, dW_t is a Brownian increment.

In[11], the mean temperature at time t takes the following form

$$x_t^m = A + Bt + C\sin(\omega t + \phi).$$

Its corresponding derivative process with respect to time t is

$$\frac{dx_t^m}{dt} = B + C\omega\cos(\omega t + \phi).$$

3.2 PDE of the OU process

Let $V(x_t, y_t, t)$ be the value of a weather option written on these variables x_t and y_t, where x_t is the average temperature y_t is the degree-day index. Additionally, we assume that there is a constant risk-free interest-rate r. The temperature cannot be traded, we need to introduce a hedging instrument H that is imperfectly correlated with the temperature, and follows a geometric Brownian motion. Thus, the model for our problem is as follows:

$$\begin{aligned} dx_t &= dx_t^m + \alpha(x_t^m - x_t)dt + \sigma_t dW_t \\ &= \mu_t dt + \sigma_t dW_t, \end{aligned}$$

$$dH_t = \mu_H H_t dt + \sigma_H H_t dZ_1,$$

$$dy_t = f(x_t, t)dt,\ dM_t = rM_t dt,$$

μ_t and σ_t is the temperature fluctuation and drift, μ_H and σ_H are the drift and volatility of the underlying assets.

Since dH_t is relared to dW_t, we can rewerite the Brownian increment dW_t as $dW_t = \rho dZ_1 + \sqrt{1-\rho^2}dZ_2$, ρ is the correlation between dZ_1 and dW_t, Z_1 and Z_2 are two standard Brownian motions, Z_1 is uncorrelated with Z_2, $E[dZ_1 dZ_2 = 0]$.

We construct a hedging portfolio comprising of an option V less Δ contracts of the imperfectly correlated asset H_t. The portfolio is financed by the sale of a bond M_t:

$$\prod{}_t = V_t - \Delta H_t - M_t,$$

The assumption is that $M_t = V_t - \Delta H_t$, at the time t. This means that in $0 < t < T$ period, no funds were added or removed from the portfolio.

$$\begin{aligned} d\prod{}_t &= dV_t - \text{Ä}dH_t - dM_t \\ &= \left(\frac{\partial V}{\partial t} + \mu_x \frac{\partial V}{\partial x} + f(x,t)\frac{\partial V}{\partial y} + \frac{1}{2}\sigma_x^2 \frac{\partial^2 V}{\partial x^2}\right)dt + \sigma_x \frac{\partial V}{\partial x} dW_t \\ &\qquad -\text{Ä}\left(\mu_H H_t dt + \sigma_H H_t dZ_1\right) - rM_t dt \\ &= \left(\frac{\partial V}{\partial t} + \mu_x \frac{\partial V}{\partial x} + f(x,t)\frac{\partial V}{\partial y} + \frac{1}{2}\sigma_x^2 \frac{\partial^2 V}{\partial x^2}\right)dt + \sigma_x \frac{\partial V}{\partial x}\left(\rho dZ_1 + \sqrt{1-\rho^2}dZ_2\right) \\ &\qquad -\Delta(\mu_H H_t dt + \sigma_H H_t dZ_1) - r(V(x,y,t) - \Delta H_t)dt \end{aligned}$$

Then, let $\Delta = \dfrac{\sigma_x \rho}{\sigma_H H_t}\dfrac{\partial V}{\partial x}$, can eliminate sources of randomness, and the coefficient of dZ_1 is zero. Thus, we can get

$$\begin{aligned} d\prod{}_t = &\left(\frac{\partial V}{\partial t} + \mu_x \frac{\partial V}{\partial x} + f(x,t)\frac{\partial V}{\partial y} + \frac{1}{2}\sigma_x^2 \frac{\partial^2 V}{\partial x^2} - rV\left(x,y,t\right) + \frac{\sigma_x}{\sigma_H}\rho r \frac{\partial V}{\partial x} - \frac{\sigma_x}{\sigma_H}\rho\mu_H \frac{\partial V}{\partial x}\right)dt \\ &+ \sqrt{1-\rho^2}\sigma_x \frac{\partial V}{\partial x} dZ_2. \end{aligned}$$

Obviously, this hedging strategy only partially hedges the portfolio, and there is still randomness in the portfolio. Therefore, we assume that the expectation of $\prod_t$ change is zero, and when the basis is the HDD index, we get a PIDE of a weather derivative, and the temperature follows the mean regression Brownian motion:

$$\frac{\partial V}{\partial t}+\gamma(x,t)\frac{\partial V}{\partial x}+\frac{1}{2}\sigma_x^2\frac{\partial^2 V}{\partial x^2}+(18-x)^+\frac{\partial V}{\partial y}-rV=0$$

Let $\gamma(x,t)=\mu_x-\frac{\sigma_x}{\sigma_H}\rho\mu_H+r\frac{\sigma_x}{\sigma_H}\rho$, and the terminal conditions meet the following formula:

$$V(x_{\min},y,t)=0,\quad \frac{\partial V}{\partial x}(x_{\max},y,t)=0,\quad \lim_{y\to\infty}V(x,y,t)=0,$$

$$\begin{cases} V(x,y,T)=tick\times(K-y_H(T))^+ & \text{put opiton,}\\ V(x,y,T)=tick\times(y_H(T)-K)^+ & \text{call option.}\end{cases}$$

3.3 The HJB equation of OU process

Because we have to consider the problems brought by the HJB equation, we know that in the market, the volatility and drift rate of geometric Brownian motion are changed, so here we give the volatility of weather derivatives to meet a range of changes, we will get the following HJB equation:

$$\frac{\partial V}{\partial t}+\max_{\sigma_x\in[\sigma_{\min},\sigma_{\max}]}\gamma(x,t)\frac{\partial V}{\partial x}+\frac{1}{2}\sigma_x^2\frac{\partial^2 V}{\partial x^2}+(18-x)^+\frac{\partial V}{\partial y}-rV=0.$$

$$\gamma(x,t)=\mu_x-\frac{\sigma_x}{\sigma_H}\rho\mu_H+r\frac{\sigma_x}{\sigma_H}\rho.$$

4. Solve the HJB equation

We let $t=T-\tau, V(x,y,t)=V(x,y,T-\tau)=U(x,y,\tau)$, thus we can get

$$-\frac{\partial V}{\partial t}=\frac{\partial U}{\partial \tau},\quad \frac{\partial V}{\partial x}=\frac{\partial U}{\partial x},\quad \frac{\partial V}{\partial y}=\frac{\partial U}{\partial y},\quad \frac{\partial^2 V}{\partial x^2}=\frac{\partial^2 U}{\partial x^2}.$$

For HDD indices, the weather derivative option pricing model becomes

$$\frac{\partial U}{\partial \tau}-(18-x)^+\frac{\partial U}{\partial y}=\gamma(x,T-\tau)\frac{\partial U}{\partial x}+\frac{1}{2}\sigma_x^2\frac{\partial^2 U}{\partial x^2}-rU,$$

$$U(x_{\min},y,\tau)=0,$$

$$\frac{\partial U}{\partial x}(x_{\max},y,\tau)=0,$$

$$\lim_{y\to\infty}U(x,y,\tau)=0,\quad U(x,y,0)=tick\times(K-y_H(0))^+,$$

Before going any further, let's introduce the following definition. We use equally spaced grids on the x coordinate to discrete $x_i=x_{\min}+i\Delta x, i=0,1,\ldots,I$, similarly, on y coordinate let $y_j=y_{\min}+j\Delta y, j=0,1,\ldots,J$. Then let $U_{i,j}^n=U(x_i,y_j,\tau_n)$ denote the solution at the average temperature node x_i for the HDD index y_j and time level n. A semi-Lagrangian method is introduced to solve the above equation by using the method in literature [14].

The Lagrangian derivative along a trajectory $\frac{dy}{d\tau}=-(18-x)^+$ is

$$\frac{DU}{D\tau} = \frac{\partial U}{\partial \tau} + \frac{\partial U}{\partial y}\frac{dy}{d\tau}.$$

Then, we can get this form

$$\frac{DU}{D\tau} = \max_{\sigma_x \in [\sigma_{\min}, \sigma_{\max}]} \gamma(x, T-\tau)\frac{\partial U}{\partial x} + \frac{1}{2}\sigma_x^2 \frac{\partial^2 U}{\partial x^2} - rU.$$

Table 1. Parameter Settings

Parameters	Values	Parameters	Values
r	0.05/365	$x_{\min}$	-50
K	{1000,100,10}	$x_{\max}$	50
λ	0.08	$y_{\min}$	0
α	0.237	$y_{\max}$	$T \times (18 - x_{\min})$
σ	$[\sigma_{\min}, \sigma_{\max}]$	x_t^m	5.97+6.57·10-5t+10.4sin(3t-2.01)
$f(x,t)$	$(18-x)^+$	*tick*	1

We modify the program to get the following image of option prices:

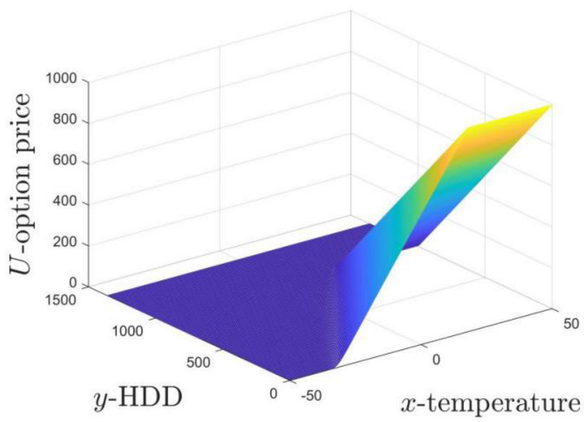

Figure 1. The variation in the solution profiles in time for a put option with T=20

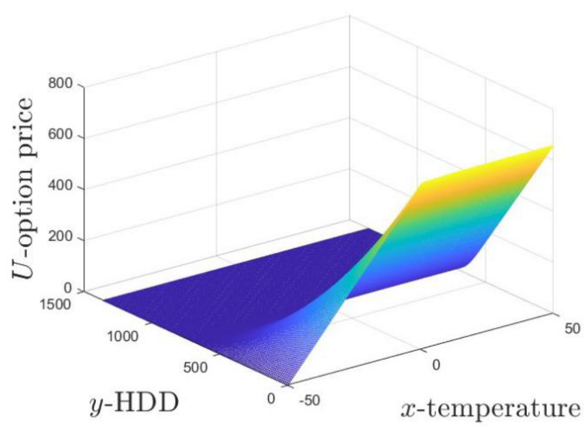

Figure 2. The variation in the solution profiles in time for a put option with T=20 in Li's paper

We can see from the figure that the overall trend of the option price picture obtained is the same as that obtained by Li, but the option price is higher. Because we're here from $[\sigma_{\min}, \sigma_{\max}]$ get the one that makes the option price maximum σ

value, It is also found that the optimal σ value obtained is always the minimum value of the set σ range.

5. Conclusion

This paper discusses the complexity and uncertainty in Asian option pricing by introducing the uncertainty volatility hypothesis and using the HJB equation. The research shows that uncertainty volatility has a significant impact on the price of Asian options, resulting in a widening of the price range of options, which reflects the increased uncertainty and risk in the market. In practical applications, traditional pricing models may underestimate risk, while the method proposed in this paper can more accurately capture volatility in the market. Overall, this study provides theoretical support for the improvement of option pricing models and emphasizes the importance of considering uncertainty when dealing with complex financial instruments. This method can provide investors and risk managers with more reliable pricing and decision-making basis under uncertain market conditions.

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