Region-Educational Research and Reviews, 2024,Volume 6, Issue 10
DOI:10.32629/rerr.v6i10.2720
ISSN Online:2661-4634
ISSN Print:2661-4626

Research on the Application of Laplace's Theorem in the Calculation of Determinants

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Abstract: As an important part of algebra, determinant can be used to judge the solution of linear equations, calculate the dimension of vector space, judge the reversibility of square matrices, etc., but its calculation is often difficult to solve. This paper first introduces Laplace's theorem and its inferences, and gives the proof process. After that, the process of solving problems with Laplace's theorem is summarized, some ideas are obtained, and the feasibility and applicability of ideas are proved by examples.

Keywords: determinant, algebra, Laplace's theorem, idea, feasibility and applicability

Introduction

The determinant plays a very important role in higher algebra. It is an important property of matrix, which can be used to judge the solution of linear equations, calculate the dimension of vector space, and judge the reversibility of square matrices. Determinants are also widely used in linear transformations and multi-variable calculus, such as when calculating Jacobian determinants to describe the relationship between partial derivatives of functions of several variables. Laplace's theorem as an important method in the calculation of determinants, here we will introduce some ideas and skills in the use of this method, to provide readers with some knowledge and understanding of this theorem.

1. The proof and generalization of Laplace's theorem

1.1 Laplace's theorem and its proof

Definition 1[1]

$$\begin{vmatrix} a_{11} \cdots & a_{1j} \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{i1} \cdots & a_{ij} \cdots & a_{in} \\ \vdots & \vdots & \vdots \\ a_{n1} \cdots & a_{nj} \cdots & a_{nn} \end{vmatrix} \quad \begin{Vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

Row i and column j of element a are removed from the determinant, and the remaining elements form a determinant of order n-1 according to the original arrangement It is called the co-child of the element, denoted as M_{ij}

Definition 2[1]

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$$\begin{vmatrix} a_{11}\cdots & a_{1j}\cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{i1}\cdots & a_{ij}\cdots & a_{in} \\ \vdots & \vdots & \vdots \\ a_{n1}\cdots & a_{nj}\cdots & a_{nn} \end{vmatrix}$$

Row i and column j of the element a are delimited in the determinant, and the remaining elements form a determinant of order n-1 according to the original arrangement method, which is called the co-factor of the element, and the product of this co-factor and this co-factor is called the algebraic co-factor of the element, which is denoted as A_{ij}

$$(-1)^{\text{i+j}}\left\|\begin{matrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{matrix}\right|$$

Lemma 1[1] Every term in the product of any sub-expression M of the determinant D and its algebraic co-sub-expression A is a term in the expansion of the determinant D, and the sign agrees.

Theorem 1[2] (Laplace's theorem)

Any row k(1≤k<n-1) is set in the determinant D. The sum of the products of all K-order subexpressions consisting of the K-row elements and their algebraic co-factors is equal to the determinant D

Proof [3]

Let the sub-formula obtained after taking row k in D be M1,M2,... Mt, whose co-factors are respectively A1,A2,... Theorem requires proof $D = M_1A_1 + M_2A_2 + \cdots + M_tA_t$ According to the lemma every term is a term in D and has the same sign. And (i≠j) has no common entries. So in order to prove the theorem, all we have to do is prove that the number of terms on both sides of the equation is equal. It's clear that the left side of the equation has terms in common, and in order to calculate the number of terms on the right, let's first figure out what we know by taking the sub-formula $\text{t}=\text{C}_n^{\text{k}} = \frac{n!}{k!(n-k)!}$ Because M has k! Item :A has (n-k)! Terms. So the right side has terms in common. Theorem proving

Theorem 2[4] The product of two n-order determinants D1 and D2 is equal to a n-order determinant C where CIJ is the sum of the product of the I-th row elements of D1 and the corresponding JTH column elements of D2, respectively, i.e

$$c_{ij} = a_{i1}b_{i1} + a_{i2}b_{i2} + \cdots + a_{in}b_{in}$$ [5]

$$D_2 = \left\|\begin{matrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{matrix}\right| \quad D_1 = \left\|\begin{matrix} \text{a}_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix}\right|$$

$$C=\begin{Vmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{Vmatrix}$$

1.2 Generalized form of theorem

Theorem 3[6] (1) determinant of order m+n

$$\begin{vmatrix} A_{\mathrm{nxn}} & 0 \\ C_{\mathrm{mxn}} & B_{\mathrm{mm}} \end{vmatrix} = |A_{\mathrm{nxn}}||B_{\mathrm{mxm}}|$$

Take the first n rows, expand, only if its principal sub-formula is $|A_{\mathrm{nn}}|$, it can not be zero. Since at least one of the other principal sub-formulas has all zeros, all the other principal sub-formulas are equal to zero. And when the principal sub-expression is $|A_{\mathrm{nn}}|$, the other sub-expressions are $|B_{\mathrm{mm}}|$, and the coefficients of the algebraic co-expressions are $(-1)^{1+\cdots+n+1+\cdots+n}=(-1)^{n(n+1)}$, whether n is odd or even, n(n+1) is even, so the corresponding algebraic co-expression of the principal sub-expression is So, according to Laplace's theorem $\begin{vmatrix} A_{\mathrm{nxn}} & 0 \\ C_{\mathrm{mxn}} & B_{\mathrm{mm}} \end{vmatrix} = |A_{\mathrm{nxn}}||B_{\mathrm{mxm}}|$

(2) determinant of order m+n

$$\begin{vmatrix} 0 & A_{\mathrm{nxn}} \\ B_{\mathrm{mxm}} & C_{\mathrm{mxn}} \end{vmatrix} = (-1)^{\mathrm{mn}} |A_{\mathrm{nxn}}||B_{\mathrm{mxm}}|$$

Proof: Take the first n rows and expand them so that they can be non-zero only if their principal sub-formula is $|A_{\mathrm{nn}}|$. Since at least one of the other principal sub-formulas has all zeros, all the other principal sub-formulas are equal to zero. And when the principal sub-expression is $|A_{\mathrm{nn}}|$, the remaining sub-expressions are $|B_{\mathrm{mm}}|$, and the co-factors of the co-expressions are $(-1)^{1+2+\cdots+n+(m+1)+\cdots+(m+n)}=(-1)^{mn+n(n+1)}$, and whether n is odd or even, n(n+1) is even, so the co-expression corresponding to the principal sub-expression is then, according to Laplace's theorem $\begin{vmatrix} 0 & A_{\mathrm{nxn}} \\ B_{\mathrm{mxm}} & C_{\mathrm{mxn}} \end{vmatrix} = (-1)^{\mathrm{mn}} |A_{\mathrm{nxn}}||B_{\mathrm{mxm}}|$

2 Applications and ideas

2.1 Ideas

Idea 1

Before using Laplace's theorem, we should use the elementary transformation of the determinant to simplify our selected row or column to produce more than one 0.

Idea 2

As much as possible, we simplify the multiple rows or columns of a determinant to a determinant that has multiple zeros in this row or column.

Train of thought

We can convert the determinant to an upper and lower triangular determinant.

Idea 3

When using Laplace's theorem, the row expansion or column expansion chosen should satisfy the following principles.

(1) When selecting a row or column, it should be preferred to choose more zeros or more identical numbers

(2) The algebraic co-factor corresponding to a number that is not 0 should be easy to compute or be quadratic Laplacian or triangulable

2.2 Examples and their solution ideas

Example Question 1 $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & -3 \\ 1 & 2 & 2 & 5 \\ 4 & 3 & 2 & 1 \end{vmatrix}$

Analysis[7]: Looking at this problem we find that the first row of the determinant in this problem is all 1 we can simplify the first row, look at the simplified determinant, we find that the second row or the second column has two identical elements, so we simplify the second row or the third row of this determinant again. And then we can think about idea number two, where we can apply Laplace's theorem to this determinant in terms of the first row and then the second row.

solution

After the initial simplification $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & -3 \\ 1 & 2 & 2 & 5 \\ 4 & 3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & -1 & -5 \\ 1 & 1 & 1 & 4 \\ 4 & -1 & -2 & -3 \end{vmatrix}$

After simplifying again

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 1 & 0 & -1 \\ 4 & -1 & -1 & 2 \end{vmatrix}$$

We expand the first row of the above determinant，We're expanding the second row of this determinant

$$\begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & -1 & 2 \end{vmatrix} = (-1)^{1+1+1} \begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix} = 1$$

Example Question 2 $\begin{vmatrix} 1 & \frac{1}{2} & 1 & 1 \\ -\frac{1}{3} & 1 & 2 & 1 \\ \frac{1}{3} & 1 & -1 & \frac{1}{2} \\ -1 & 1 & 0 & \frac{1}{2} \end{vmatrix}$

Analysis[8]: Observing this problem, we find that the first row and the second column of the determinant in this problem have three ones. We may wish to simplify the first row, observe the simplified determinant, we find that the second row has two identical elements, so we simplify the second row of this determinant again. And then we can think about idea number two, where we can apply Laplace's theorem to this determinant in terms of the first row and then the second row.

solution

After the initial simplification $\begin{vmatrix} 1 & \frac{1}{2} & 1 & 1 \\ -\frac{1}{3} & 1 & 2 & 1 \\ \frac{1}{3} & 1 & -1 & \frac{1}{2} \\ -1 & 1 & 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{7}{3} & \frac{7}{3} & \frac{4}{3} \\ \frac{1}{3} & \frac{5}{3} & -\frac{4}{3} & \frac{1}{6} \\ -1 & 3 & 1 & \frac{3}{2} \end{vmatrix}$

After simplifying again

$$\frac{1}{2}\begin{vmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{7}{3} & \frac{7}{3} & \frac{4}{3} \\ \frac{1}{3} & \frac{5}{3} & -\frac{4}{3} & \frac{1}{6} \\ -1 & 3 & 1 & \frac{3}{2} \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{7}{3} & 0 & 0 \\ \frac{1}{3} & \frac{5}{3} & -3 & -\frac{33}{42} \\ -1 & 3 & -2 & -\frac{3}{14} \end{vmatrix}$$

We expand the first row of the above determinant

$$\frac{1}{2}\begin{vmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{7}{3} & 0 & 0 \\ \frac{1}{3} & \frac{5}{3} & -3 & -\frac{33}{42} \\ -1 & 3 & -2 & -\frac{3}{14} \end{vmatrix} = \frac{(-1)^{1+1}}{2}\begin{vmatrix} \frac{7}{3} & 0 & 0 \\ \frac{5}{3} & -3 & -\frac{33}{42} \\ 3 & -2 & -\frac{3}{14} \end{vmatrix} = \frac{1}{2}\begin{vmatrix} \frac{7}{3} & 0 & 0 \\ \frac{5}{3} & -3 & -\frac{37}{42} \\ 3 & -2 & -\frac{3}{14} \end{vmatrix}$$

We're expanding the second row of this determinant

$$\frac{1}{2}\begin{vmatrix} \frac{7}{3} & 0 & 0 \\ \frac{5}{3} & -3 & -\frac{37}{42} \\ 3 & -2 & \frac{3}{14} \end{vmatrix} = \frac{7}{6}\begin{vmatrix} -3 & -\frac{37}{42} \\ -2 & \frac{3}{14} \end{vmatrix} = -\frac{13}{12}$$

Example Question 3 $\begin{Vmatrix} x & y & 0 & \cdots & 0 & 0 \\ 0 & x & y & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & y \\ y & 0 & 0 & \cdots & 0 & x \end{Vmatrix}$

Analysis[9]: Observing this problem, we find that the determinant in this problem already satisfies the first idea, so we can directly consider the second idea, observing this determinant, we find that every row or every column has the same situation, so which row is selected is consistent.

Solution:

We expand the first column, using Laplace's theorem

$$\left\|\begin{matrix} x & \mathrm{y} & 0 & \cdots & 0 & 0 \\ 0 & x & y & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & y \\ y & 0 & 0 & \cdots & 0 & x \end{matrix}\right| = (-1)^{1+1} x \left\|\begin{matrix} x & y & \cdots & 0 & 0 \\ 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & x & y \\ 0 & 0 & \cdots & 0 & x \end{matrix}\right| + (-1)^{1+n} y \left\|\begin{matrix} y & 0 & \cdots & 0 & 0 \\ x & y & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & y & x \\ 0 & 0 & \cdots & 0 & y \end{matrix}\right| = x^n - (-1)^n y^n$$

Example Question 4 $\left\|\begin{matrix} b_1 & b_2 & b_2 & \cdots & b_2 \\ b_2 & b_2 & b_2 & \cdots & b_2 \\ b_2 & b_2 & b_3 & \cdots & b_2 \\ \vdots & \vdots & \vdots & & \vdots \\ b_2 & b_2 & b_2 & \cdots & b_n \end{matrix}\right|$

Analysis[10]: Observing this problem, we find that the second row of the determinant in this problem has many identical numbers that can be simplified, and we consider the second thought, observing the simplified determinant, we find that except for the second row, there is only one number in each row that is not 0, so we may choose the first row to expand.

solution

Simplify to $\left\|\begin{matrix} b_1-b_2 & 0 & 0 & \cdots & 0 \\ b_2 & b_2 & b_2 & \cdots & b_2 \\ 0 & 0 & b_3-b_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & b_n-b_2 \end{matrix}\right|$ Expand by the first line

$$\left\|\begin{matrix} b_1-b_2 & 0 & 0 & \cdots & 0 \\ b_2 & b_2 & b_2 & \cdots & b_2 \\ 0 & 0 & b_3-b_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & b_n-b_2 \end{matrix}\right| = (-1)^{1+1} (b_1-b_2) \begin{vmatrix} b_2 & b_2 & \cdots & b_2 \\ 0 & b_3-b_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b_n-b_2 \end{vmatrix} = b_2 (b_1-b_2) \prod_{i=3}^{n} (b_i - b$$

3. Conclusion

When we're dealing with the calculation of a determinant, the first thing we should do is look at the determinant, and if we're thinking about using Laplace's theorem to solve this calculation of a determinant, we should use the elementary variation of a row or column, and we should try to reduce the multiple rows or columns of the determinant as much as possible to the determinant that has multiple zeros in this row or column. If it can be turned into an upper and lower triangular determinant, we will turn it into a triangular determinant, and after the solution is complete, we will expand by row and column according to idea 2 to calculate. In the application of Laplace's theorem, if it cannot be simplified after one simplification, we should refer to idea 3 again for quadratic simplification. At the same time, if the value of the determinant cannot be calculated after one expansion by row and column, we should also carry out quadratic expansion by row and column to transform the higher-order complex determinant into multiple low-order simple determinant solutions. The idea mentioned in this topic is only. It still takes a lot of practice to compute a determinant quickly.

4. Closing remarks

Laplace's theorem is a generalization of row by column expansion, and row by column expansion is a special case of

Laplace's theorem, and the two are mutually reinforcing and inseparable. In addition, the main principle of Laplace's theorem is to transform the higher-order determinant into a few lower-order determinant solution, the complex determinant into a few simple determinant calculation, this idea is an important idea in algebra, so learning Laplace's theorem can help you learn the real algebra, into the door of mathematics. It is hoped that readers can attach great importance to the study and use of this simple theorem.

Conflicts of interest

The author declares no conflicts of interest regarding the publication of this paper.

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