# Research on Periodic Solutions of Non-autonomous Second-order Hamiltonian Systems

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**Abstract:** In the field of nonlinear science, Hamiltonian systems are crucial models extensively applied in engineering, physics, and biology. The existence of periodic solutions in non-autonomous second-order Hamiltonian systems, as a relatively specialized form, has been a focal and challenging research area. This paper employs variational methods to analyze the existence of periodic solutions in such systems, aiming to provide insights for related research endeavors.
***Keywords*:** second-order non-autonomous hamiltonian systems; periodic solutions; variational methods; principle of least action; local surround theorem; saddle point theorem; mountain pass lemma

## 1. Introduction

In the backdrop of advancing aerospace technology and artificial celestial bodies, exploring periodic solutions of second-order non-autonomous Hamiltonian systems has become crucial and significant. The system primarily investigates and describes the dynamic behavior of objects in time-varying force fields, aiming to predict long-term system behaviors and stability [1]. Previous research on such issues has heavily relied on numerical methods and approximate theories, which often prove inadequate when dealing with complex systems and high demands for accuracy [2]. Thus, the adoption of analytical methods for identifying periodic solutions has emerged as an important trend in this research field.

## 2. Overview of Theoretical Foundations

### 2.1 Concept and Characteristics of Hamiltonian Systems

Hamiltonian systems represent a significant formulation in classical mechanics, named after the Irish mathematician and physicist William Rowan Hamilton. In Hamiltonian systems, the dynamics of physical systems are described using a Hamiltonian function, which is the sum of the system's potential and kinetic energies [3]. The principle of Hamiltonian mechanics forms the basis of the system's equations of motion, indicating that the Hamiltonian action attains extremum values over fixed time intervals during actual motion. In mathematical terms, Hamiltonian systems are expressed as:

$$\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = \frac{\partial H}{\partial q} \tag{1}$$

In equation (1), $\dot{q}$ represents the derivative of the generalized coordinate q with respect to time, indicating the rate of change of q over time. $\dot{q}$ denotes the derivative of the generalized momentum p with respect to time, indicating the rate of change of p. $\frac{\partial H}{\partial p}$ represents the partial derivative of the Hamiltonian H with respect to p, and $\frac{\partial H}{\partial q}$ represents the partial derivative of the Hamiltonian H with respect to q.In general, the Hamiltonian H is a function of the generalized coordinate q and the generalized momentum p, expressed as:

$$H = H(q, p) \tag{2}$$

Energy conservation is a crucial characteristic of Hamiltonian systems, stating that the Hamiltonian function remains constant along the system's trajectories for closed systems. Liouville's theorem asserts that the volume of phase space remains invariant, further underlining the special status of Hamiltonian systems in statistical mechanics and quantum mechanics [4].

### 2.2 Overview of Variational Calculus and Its Application

The study of extremum problems for functionals relies on variational calculus, a critical mathematical method. In fields

such as physics and engineering, many problems, including the brachistochrone problem and shortest path problems, can be summarized as finding extremal values of functionals [5]. Applying variational calculus to study Hamiltonian systems allows us to seek the system's actual trajectory corresponding to extremal paths of the action. Specifically, in Hamiltonian systems, the action is defined as:

$$S = \int_{t_1}^{t_2} L\left(q, \dot{q}, t\right) dt \tag{3}$$

In equation (3), S describes the dynamics of the system between times $t_1$ and $t_2$. $L\left(q, \dot{q}, t\right)$ represents the Lagrangian, typically given by the difference between kinetic and potential energies:

$$L = T - V \tag{4}$$

The fundamental idea of variational calculus is to transform extremum problems of functionals into solving Euler-Lagrange equations. For the action S, its corresponding Euler-Lagrange equation is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{5}$$

### 2.3 Related Mathematical Tools and Theorems

As a fundamental principle in physics, the principle of least action reveals the true trajectory of Hamiltonian systems over a fixed period, achieving extremal values of the Hamiltonian action. This principle involves variational calculations of the action, employing complex mathematical computations to derive the system's equations of motion and deepen our understanding of physical phenomena. The core idea of the local surround theorem asserts the existence, under specific conditions, of a closed trajectory where the action achieves a local extremum. In practical applications, this requires analyzing the potential energy function, kinetic energy function, and precisely defining periodic solutions of the system to accurately understand the dynamic characteristics of Hamiltonian systems [6]. In functional analysis, the saddle point theorem identifies solutions that are not necessarily global minima but still belong to the system's true motion trajectory, involving a detailed analysis of second-order variations of the action [7].

## 3. System Model and Assumptions

### 3.1 Non-autonomous Second-order Hamiltonian Model

In the non-autonomous second-order Hamiltonian system, the dynamics are described fundamentally by a nonlinear second-order differential equation, given by the system equation:

$$\ddot{q} + V'\left(q\right) = f\left(t\right) \tag{6}$$

In equation (6), $\ddot{q}$ represents the generalized coordinate acceleration, $V'\left(q\right)$ denotes the derivative of the potential energy function $V\left(q\right)$ with respect to q, reflecting the influence of the potential gradient on the system dynamics. f(t) represents the time-dependent external force term, which disrupts the system's closed and autonomous nature.

To delve deeper into the analysis of the non-autonomous second-order Hamiltonian model, it is essential to construct the Hamiltonian function and the action function. The action function is defined as in equation (3). For the non-autonomous second-order Hamiltonian system, the Lagrangian can be expressed as:

$$L\left(q, \dot{q}, t\right) = T\left(\dot{q}\right) - V\left(q\right) + q \cdot f\left(t\right) \tag{7}$$

In equation (7), $T\left(\dot{q}\right)$ represents the system's kinetic energy, expressed as a function of $\frac{1}{2}m\dot{q}^2$. Substituting equation (7) into equation (3) yields:

$$S = \int_{t_1}^{t_2} \left(\frac{1}{2}m\dot{q}^2 - V\left(q\right) + q \cdot f\left(t\right)\right) dt \tag{8}$$

The action function provides feedback on the total "cost" of the system over the time interval. According to the principle of least action in physics, the true physical path minimizes this action function. To find this path, we vary the action function and set it to zero:

$$\delta S = \int_{t_1}^{t_2} \left( m\dot{q}\delta\dot{q} - V'(q)\delta q + \delta q \cdot f(t) \right) dt = 0 \tag{9}$$

Solving equation (9) allows us to determine the system's true trajectory. Furthermore, exploring the properties of the potential energy function and the external force term enhances our understanding of the system's dynamic behavior. For example, analyzing the shape of the potential energy function and the positions of its minima can reveal whether the system is in a stable state. Investigating the periodicity and amplitude of the external force f(t) helps us understand the system's non-autonomous nature.

### 3.2 System Functions and Matrix Assumptions

In physics, the potential energy function describes the potential energy of a system under different states, allowing for an accurate analysis of the system's behavior. In research and analysis, it is assumed that the potential energy function $V(q)$ remains continuously operational and meets the standard of smoothness. This implies that all generalized coordinates $q$ and the potential energy function $V(q)$ have definite values that vary with changes in generalized coordinates. Smoothness further requires that the potential energy function be both differentiable and continuous.From a mathematical perspective, $V(q)$ belongs to the class $C^n$, where n is a sufficiently large integer, ensuring that it can be differentiated multiple times [8].

The external force $f(t)$ term represents the time-dependent force exerted on the system from outside. To simplify the complexity during analysis and capture the main characteristics quickly, assumed $f(t)$ is a continuous and periodic function with a period T. This assumption means that for any time t, $f(t+T) = f(t)$. This assumption is reasonable because phenomena like alternating current signals, vibrations in engineering applications, and natural occurrences often exhibit periodic behavior.

In special cases, to simplify the analysis of complex nonlinear systems, one can linearize the system around an equilibrium point using the idea of Taylor series expansion, retaining only the first-order terms to approximate the original nonlinear system. During the process of linearization, it is essential to incorporate the Jacobian matrix J, where the elements of J are the partial derivatives of the potential energy function $V(q)$ with respect to the generalized coordinates q. Near the equilibrium point $q_0$, the system can be approximated as:

$$\delta\dot{q} = J(q_0\delta q) \tag{10}$$

Here, $\delta q = q - q_0$ represents the small deviation from the equilibrium point, and $J(q_0)$ is the Jacobian matrix evaluated at the equilibrium point $q_0$.

## 4. Analysis of the Existence of Periodic Solutions

### 4.1 Discussion on the Conditions for the Existence of Periodic Solutions

To delve into the conditions for the existence of periodic solutions, it is necessary to explore from various perspectives such as the action functional, periodic boundary conditions, and the properties of the potential energy function. Primarily, in the analysis, the method of variations is employed to investigate the existence of periodic solutions, focusing on identifying paths that extremize the action functional S. During this process, the variation of the action functional is $\delta$ S=0. Specifically, for any small path variation $\delta q$, the variation of the action can be expressed as:

$$\delta S = \int_{t_0}^{t_0+T} \left( \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\delta\dot{q}}\delta\dot{q} \right) dt \tag{11}$$

Expanding this formula through a series of mathematical transformations yields the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\delta \dot{q}}\right)-\frac{\partial L}{\partial q}=0 \tag{12}$$

This equation describes the actual behavior of the system and is crucial in finding periodic solutions.

Given the focus on periodic solutions, it is crucial during analysis to emphasize the periodic boundary conditions. This condition stipulates that after a period T, the system's state should return to its initial state. For a generalized coordinate q, this condition is expressed as:

$$q\left(t_0\right)=q\left(t_0+T\right) \tag{13}$$

Periodic boundary conditions are essential for determining periodic solutions, ensuring that the system's behavior within one period remains closed.

The existence of periodic solutions is closely tied to the shape of the potential energy function. In other words, the form of the potential energy function plays a decisive role in determining the equilibrium points of the system. For instance, when the potential energy function has multiple local minima, the system tends to exhibit stable periodic solutions around each of these minima. Furthermore, the nature of periodic solutions is also influenced by the symmetry of the potential energy function. If the potential energy function exhibits symmetry around a point, it indicates the existence of symmetric periodic solutions in the system. This simplifies the problem and provides an intuitive understanding of the system's dynamic behavior. In the Lagrangian, $q\cdot f\left(t\right)$ describes the interaction of the system with the external periodic force f(t). If the period of the external force is close to the natural period of the system, it can lead to resonance phenomena. Solving the Euler-Lagrange equations analytically in practical applications is challenging, necessitating the use of numerical methods such as finite difference methods and the Runge-Kutta method to simulate the system's dynamic behavior and approximate periodic solutions.

## 4.2 Derivation and Proof of Sufficient Conditions

In physics, problems like the brachistochrone problem and the path of light propagation can be formulated as extremum problems for functionals. In the context of variational calculus, the Euler-Lagrange equation is a core concept, describing the necessary conditions for a function to extremize a functional. According to the Euler-Lagrange equation, the system must satisfy Equation (12). For a simple case, the kinetic energy T is expressed as:

$$T=\frac{1}{2}m\dot{q}^2 \tag{14}$$

The potential energy V is given by:

$$V=V\left(q\right) \tag{15}$$

Substituting Equations (14) and (15) into Equation (7), we obtain:

$$L=\frac{1}{2}m\dot{q}^2-V\left(q\right) \tag{16}$$

Next, applying the variational method to the Lagrangian, we derive the Euler-Lagrange equation. Substituting Equation (16), we get:

$$\frac{d}{dt}\left(m\dot{q}\right)-\frac{\partial V}{\partial q}=0 \tag{17}$$

Simplifying further results in:

$$m\ddot{q}+\frac{\partial V}{\partial q}=0 \tag{18}$$

The minima of the potential energy function correspond to the stable states of the system. If the potential energy function has multiple minima with suitable energy differences between them, the system may exhibit periodic motion between these

minima. Moreover, if the potential energy function exhibits certain symmetries, the motion of the system will reflect these symmetries, thereby supporting the inference of the existence of periodic solutions.

In the nonlinear analysis, the local surround theorem and saddle point theorem are important tools, involving analysis of the potential energy function and external force terms to determine the existence of periodic solutions under specific conditions. For example, if the potential energy function has a local minimum in a certain region and the external force terms satisfy conditions such as periodicity or boundedness, these theorems can be used to prove the existence of periodic solutions. Specifically, the local surround theorem identifies paths around the local minima of the potential energy function, ensuring periodic motion of the system along these paths. The saddle point theorem helps to determine the stability and uniqueness of periodic motion during the study.

Analytical methods sometimes cannot directly prove the existence of periodic solutions, especially when the system is complex or involves nonlinear terms, which can pose significant challenges. In such cases, numerical methods and computer simulations can be employed to simulate and analyze the existence of periodic solutions. By iteratively computing to approximate the true solution of the system, such as using the Runge-Kutta method to solve the Euler-Lagrange equation, one can obtain the system's motion trajectory and determine whether periodic motion exists.

## 5. Conclusion

This paper provides a brief overview of the fundamental concepts and characteristics of Hamiltonian systems, and discusses the application of variational methods and related mathematical tools. Starting from the perspective of constructing non-autonomous second-order Hamiltonian models, it explores hypotheses regarding system functions and matrices, laying the theoretical and practical foundations for subsequent research. When analyzing the existence of periodic solutions, the paper comprehensively discusses the action function, periodic boundary conditions, and the properties of the potential energy function. It derives conditions for the existence of periodic solutions using the Euler-Lagrange equation, and discusses how the shape and symmetry of the potential energy function influence the existence of periodic solutions, as well as how the periodicity and magnitude of external forces affect the non-autonomous nature of the system. The aim is to provide a comprehensive theoretical framework and analytical methods for the study of periodic solutions in non-autonomous second-order Hamiltonian systems.

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