A Class of Pyramidal Central Configurations with Logarithmic Potentials

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Abstract: It is well known that for the pyramidal central configuration of the five-body problem, four masses are located at the vertices of the square and the fifth mass is located on a line perpendicular to the plane containing the square. And the line passes through the geometric center of the square. If the potential is Newtonian, then the values of the four masses at the vertices of the square are equal. In this paper, by using some properties of circulant matrices, we find that if the potentials are logarithmic potentials, then the values of the four masses are equal, too.
Keywords: n-body problem, pyramidal central configurations, logarithmic potentials, circulant matrices

1. Introduction

In n-body problem, central configurations play a significant role in the analysis of collision

orbits, expanding gravitational systems, and the limitations that affect the configurations assumed by the bounded motion [1]. For the Newtonian potential, in 1772, Lagrange [2] discovered the famous Lagrange equilateral-triangle central configuration, where three masses are located at the vertices of the regular 3-polygon, and showed that the values of the three masses may not be equal. In 1985, also for the Newtonian potential, for regular n-polygon central configuration with n⩾4, Perko and Walter [3] proved a surprising result that if n masses are located at the vertices of regular n-polygon, then the values of the n masses must be equal. In 2019, Wang [4] extended the result to general homogeneous potentials.

For the spatial n-body problem, when the potential is the Newtonian potential, a regular n+1 polyhedron where the base is a regular n-polygon, forms a pyramidal central configuration [5].

Fayçal [6] investigated the pyramidal central configuration of the 5-body problem where four masses are located at the vertices of a rectangle and the potential is Newtonian potential, and proved that the rectangle must be a square. In this paper, we investigate the spatial 5-body problem with the potentials are logarithmic potentials, which are the limiting cases of general homogeneous potentials. More precisely, suppose that four masses are located at the vertices of a square and the fifth mass is located on a straight line perpendicular to the plane that contains the square, and the straight line passes through the geometric center of the square, we investigate the mass values in the pyramidal central configuration where the base is a square and the potential are logarithmic potentials.

2. Challenges

References [7] show us that for the masses $m_1, m_2, \cdots, m_5 \in R^+$ with the corresponding positions $q_1, q_2, \cdots, q_5 \in (R^3)^5$, the pyramidal central configuration exist if and only if there exists constant $\lambda \in R$ such that the following equations (1.1) and (1.2) hold:

$$\sum_{j=1, j\neq k}^{5} \frac{m_j m_k}{\left|q_j - q_k\right|^2}(q_j - q_k) = -\lambda m_k (q_k - c_0),\ 1 \le k \le 5, \tag{1.1}$$

where $q = (q_1, q_2, \cdots, q_5) \in \cup_{1\le k\le 5}\left\{(q_1, q_2, \cdots, q_5) \in (R^3)^5 : q_i \neq q_j\right\}$,

$$\begin{cases} q_l = (\rho_l, 0) = (e^{l\pi i}, 0)\ with\ l \in \{1,2\},\ \ \rho_k = e^{k\pi i}\ with\ k \in Z, \\ q_d = (\rho_d e^{\frac{1}{2}\pi i}, 0) = (e^{(d+\frac{1}{2})\pi i}, 0)\ with\ d \in \{3,4\},\ \ q_5 = (0+0i, h), \\ c_0 = (\hat{c}_0, h_0),\ \ c_0 = \dfrac{\sum_{1\le l\le 2} m_l \rho_l + \sum_{3\le d\le 4} m_d \rho_d e^{\frac{1}{2}\pi i}}{\sum_{1\le k\le 5} m_k},\ \ h_0 = \dfrac{m_5 h}{\sum_{1\le k\le 5} m_k}, \end{cases} \tag{1.2}$$

the barycenter of the five body system is $c_0 = [\sum_{1\le k\le 5} m_k q_k]/[\sum_{1\le k\le 5} m_k]$, and $h>0$ is the distance from q_5 to the plane that contains the square. When the potentials are logarithmic potentials, which are diffenrent from the Newtonian potential, it is difficult to obtain relationships among mass values from equations (1.1) and (1.2).

3. Method

Our method is define the five 2×2 circulant matrices A, B, D, B^* and D^* as follows:

$$A=\begin{pmatrix} 0 & \frac{1-\rho_1}{|1-\rho_1|^2} \\ \frac{1-\rho_1}{|1-\rho_1|^2} & 0 \end{pmatrix}, B=\begin{pmatrix} \frac{1}{\left|1-\rho_0 e^{\frac{1}{2}\pi i}\right|^2} & \frac{1}{\left|1-\rho_1 e^{\frac{1}{2}\pi i}\right|^2} \\ \frac{1}{\left|1-\rho_1 e^{\frac{1}{2}\pi i}\right|^2} & \frac{1}{\left|1-\rho_0 e^{\frac{1}{2}\pi i}\right|^2} \end{pmatrix}, D=\begin{pmatrix} \frac{\rho_0}{\left|1-\rho_0 e^{\frac{1}{2}\pi i}\right|^2} & \frac{\rho_1}{\left|1-\rho_1 e^{\frac{1}{2}\pi i}\right|^2} \\ \frac{\rho_1}{\left|1-\rho_1 e^{\frac{1}{2}\pi i}\right|^2} & \frac{\rho_0}{\left|1-\rho_0 e^{\frac{1}{2}\pi i}\right|^2} \end{pmatrix},$$

$$B^*=\begin{pmatrix} \frac{1}{\left|e^{\frac{\pi}{2}i}-\rho_0\right|} & \frac{1}{\left|e^{\frac{\pi}{2}i}-\rho_1\right|} \\ \frac{1}{\left|e^{\frac{\pi}{2}i}-\rho_1\right|} & \frac{1}{\left|e^{\frac{\pi}{2}i}-\rho_0\right|} \end{pmatrix}, D^*=\begin{pmatrix} \frac{\rho_0}{\left|e^{\frac{\pi}{2}i}-\rho_0\right|} & \frac{\rho_1}{\left|e^{\frac{\pi}{2}i}-\rho_1\right|} \\ \frac{\rho_1}{\left|e^{\frac{\pi}{2}i}-\rho_1\right|} & \frac{\rho_0}{\left|e^{\frac{\pi}{2}i}-\rho_0\right|} \end{pmatrix}.$$

Then, we analysis the relationship between their eigenvalues and the corresponding eigenvectors, and the details are as follows.

Inserting (1.2) into (1.1), we have

$$\begin{cases} Am+(B-e^{\frac{1}{2}\pi i}D)M+\frac{m_5}{1+h^2}v_1=\lambda(v_1-\hat{c}_0 v_2), \\ (e^{\frac{1}{2}\pi i}B^*-D^*)m+e^{\frac{1}{2}\pi i}AM+e^{\frac{1}{2}\pi i}\frac{m_5}{1+h^2}v_1=\lambda(e^{\frac{1}{2}\pi i}v_1-\hat{c}_0 v_2), \end{cases} \tag{1.3}$$

where $m=(m_1,m_2)^T$, $M=(m_3,m_4)^T$, $v_1=(1,1)^T$ and $v_2=(\rho_1,\rho_1^2)^T=(-1,1)^T$. Note that there exist four constants $\alpha_1,\alpha_2,\beta_1,\beta_2\in C$ such that

$$m=\alpha_1 v_1+\alpha_2 v_2 \quad \text{and} \quad M=\beta_1 v_1+\beta_2 v_2. \tag{1.4}$$

Then, inserting (1.4) into (1.3), we can conclude that

$$\begin{cases} A(\alpha_1 v_1+\alpha_2 v_2)+(B-e^{\frac{1}{2}\pi i}D)(\beta_1 v_1+\beta_2 v_2)+\frac{m_5}{1+h^2}v_1=\lambda(v_1-\hat{c}_0 v_2), \\ (e^{\frac{1}{2}\pi i}B^*-D^*)(\alpha_1 v_1+\alpha_2 v_2)+e^{\frac{1}{2}\pi i}A(\beta_1 v_1+\beta_2 v_2)+e^{\frac{1}{2}\pi i}\frac{m_5}{1+h^2}v_1=\lambda(e^{\frac{1}{2}\pi i}v_1-\hat{c}_0 v_2). \end{cases} \tag{1.5}$$

By properties of circulant matrices [8] and direct computation, the eigenvalues of five circulant matrices $A, B, \hat{D}$, B^* and D^* are

$\mu_1(A)=\mu_2(A)=\frac{1}{2}, \mu_1(B)=\mu_1(B^*)=1, \mu_2(B)=\mu_2(B^*)=0, \mu_1(D)=\mu_1(D^*)=0$ and

$\mu_2(D)=\mu_2(D^*)=1$. Then, combining (1.5), we have

$$\begin{cases} \alpha_1 + 2\beta_1 + \dfrac{2m_5}{1+h^2} = 2\lambda, \ 2\alpha_1 e^{\frac{1}{2}\pi i} + \beta_1 e^{\frac{1}{2}\pi i} + \dfrac{2m_5}{1+h^2} e^{\frac{1}{2}\pi i} = 2\lambda e^{\frac{1}{2}\pi i}, \\ \alpha_2 - 2\beta_2 e^{\frac{1}{2}\pi i} = -2\lambda \hat{c}_0, \ -2\alpha_2 + \beta_2 e^{\frac{1}{2}\pi i} + \dfrac{2m_5}{1+h^2} e^{\frac{1}{2}\pi i} = -2\lambda c_0, \end{cases}$$ which implies that

$\alpha_1 + 2\beta_1 = 2\alpha_1 + \beta_1$, $\alpha_2 - 2\beta_2 i = -2\alpha_2 + \beta$ Thus, $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2 i$. Then, combining (1.4), $v_1 = (1,1)^T$ and $v_2 = (-1,1)^T$, we obtain

$$m = \alpha_1 v_1 + i\beta_2 v_2 \text{ and } M = \alpha_1 v_1 + \beta_2 v_2, \tag{1.6}$$

where $m = (m_1, m_2)^T$ and $M = (m_3, m_4)^T$. Hence $\begin{pmatrix} m_1 - m_3 \\ m_2 - m_4 \end{pmatrix} = \begin{pmatrix} -i\beta_2 + \beta_2 \\ i\beta_2 - \beta_2 \end{pmatrix}$ Thus, $\beta_2 - \beta_2 \in R$

Denote $\beta_2 = \gamma_1 + i\gamma_2$ where $\gamma_1, \gamma_2 \in R$. Then, $i\beta_2 - \beta_2 = -2\gamma_1 - 2\gamma_2 i \in R$, which implies that $\gamma_2 = 0$. Therefore, $\beta_2 = \gamma_1 \in R$. Moreover, if $\beta_2 \neq 0$, then employing $\begin{pmatrix} m_1 - m_3 \\ m_2 - m_4 \end{pmatrix} = \begin{pmatrix} -i\beta_2 + \beta_2 \\ i\beta_2 - \beta_2 \end{pmatrix}$, we have $m_2 - m_4 = i\beta_2 - \beta_2 \notin R$, and it is impossible. So $\beta_2 \neq 0$ can not occur, i.e., $\beta_2 = 0$. Then with the aid of (1.6), our conclusion holds.

4. Conclusion

Let the configuration $q = [q_1, q_2, \cdots, q_5]$ be defined as in equation (1.2). If q is a pyramidal central configuration with logarithmic potentials for the mass vector $m = [m_1, m_2, \cdots, m_5]$, then we have $m_1 = m_2 = m_3 = m_4$.

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