Region - Educational Research and Reviews, 2024, Volume6, Issue3
DOI: 10.32629/rerr.v6i3.1829
ISSN Online: 2661-4634
ISSN Print: 2661-4626

An equivalent infinitesimal substitution problem of indeterminate limit from a literature and its correction

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Abstract: This article delves into the complexities of indeterminate limits and the pivotal role of infinitesimal substitution in their resolution. Within the realm of mathematical analysis, indeterminate limits--characterized by expressions such as "0/0" and "∞/∞"--pose a significant challenge to traditional limit computation techniques. Equivalent infinitesimal substitution is applied to reduce the difficulty of calculation in solving indeterminate limit problems. However, the substitution cannot be directly applied to indeterminate infinitesimal subtraction and addition. This paper points out the problem of equivalent infinitesimal substitution in the indeterminate limit calculation which includes addition and subtraction used in many papers, and amends the infinitesimal substitution theorem given in the papers.

Key words: infinitesimal; equivalent infinitesimal substitution; indeterminate limit

1 Introduction

In the study of the function's asymptotic features, the limits of addition and subtraction must be calculated, except for the limits of multiplication and division. These limit types can be "0/0", "$\frac{\infty}{\infty}$", "$\infty-\infty$", or "$0\cdot\infty$", which are collectively referred to as indeterminate forms. The challenges associated with indeterminate forms were recognized as a significant obstacle in the development of calculus, as they often arose when trying to calculate derivatives and integrals.

Indeterminate limits can contain infinitesimal and infinite quantities. Limits of infinity can be transformed into the reciprocal of infinitesimal limits, thus the primary focus of extensive indeterminate limit research is the calculation of limits related to infinitesimal quantities. Equivalent infinitesimal substitution was significantly used to bypass the challenges of indeterminate limits. However, most college math textbooks typically only provide equivalent infinitesimal substitution theorems for multiplication and division limit calculations. The use of the substitution can't be directly employed to limits of addition. Therefore, many literatures have explored equivalent infinitesimal substitution beyond multiplication and division, such as variable limit integrals, trigonometric and inverse trigonometric functions , logarithms and exponents, etc. Literatures have provided theorems and examples of equivalent infinitesimal substitution in the calculation of sum and difference limits, but the theorems and examples used require correction, which this paper completes [1-8].

2 Theorem and correction of equivalent infinitesimal substitution in addition and subtraction

The theorem of equivalent infinitesimal substitution in multiplication and division posits that if, in a certain limit process, $\alpha\sim\overline{\alpha}$, $\beta\sim\overline{\beta}$, and if $\lim\frac{\overline{\alpha}}{\overline{\beta}}$ exists , then [9]

$$\lim\frac{\alpha}{\beta}=\lim\frac{\overline{\alpha}}{\overline{\beta}}$$

This theorem underscores the necessity of the existence that in a certain limit process, not only $\alpha\sim\overline{\alpha}$, $\beta\sim\overline{\beta}$ but also the limit of $\frac{\overline{\alpha}}{\overline{\beta}}$ must exist. This theorem indicates that equivalent infinitesimal substitution can be used to calculate limits of division and multiplication, such as $\lim\limits_{x\to 0}\left(\sqrt{1+x}-1\right)\arcsin x=\lim\limits_{x\to 0}\frac{1}{2}x\cdot x=0$. In this example, equivalent infinitesimal substitution is used in multiplication to simplify the calculation. However, sometimes it is also desired to directly use equivalent infinitesimal substitution in the limit calculation of infinitesimal addition and subtraction.

Example 1: Calculate the limit $\lim\limits_{x\to 0}\frac{\tan 2x-\sin 2x}{x\left(\sqrt{1+x^2}-1\right)}$.

Solution:

$$\lim_{x\to 0}\frac{\tan 2x-\sin 2x}{x\left(\sqrt{1+x^2}-1\right)}=\lim_{x\to 0}\frac{2x-2x}{x\cdot\frac{1}{2}x^2}=0$$

This is an incorrect solution. In general, equivalent infinitesimal substitution cannot be directly used in addition and subtraction. The correct solution to this problem is:

$$\lim_{x\to 0}\frac{\tan 2x-\sin 2x}{x\left(\sqrt{1+x^2}-1\right)}=\lim_{x\to 0}\frac{\tan 2x\,(1-\cos 2x)}{x\left(\sqrt{1+x^2}-1\right)}=\lim_{x\to 0}\frac{2x\cdot\frac{1}{2}\cdot 4x^2}{x\cdot\frac{1}{2}x^2}=8$$

To find the conditions for directly using equivalent infinitesimal substitution in addition and subtraction, some conclusions and their applications have been given in the literature. Literature provides a theorem, and the theorems given in literature directly use the theorem presented in the literature above, which is

Theorem 7 : If $\alpha\sim\alpha'$, $\beta\sim\beta'$, when $\lim\frac{\beta}{\alpha}=k\neq 1$, then $\alpha-\beta\sim\alpha'-\beta'$; when the $\lim\frac{\beta}{\alpha}=k\neq -1$, then $\alpha+\beta\sim\alpha'+\beta'$[1-5].

But the theorem was found not rigorous for a related theorem in literature [9]. Consider the following limit calculation example.

Example 2 [10]: Calculate the limit:

$$\lim_{x\to 0}\frac{\cos x-e^{\frac{x^2}{2}}}{x^2[x+\ln(1-x)]}$$

Solution 1:

$$\lim_{x\to 0}\frac{\cos x-e^{\frac{x^2}{2}}}{x^2[x+\ln(1-x)]}=\lim_{x\to 0}\frac{\cos x-1-\left(e^{\frac{x^2}{2}}-1\right)}{x^2[x+\ln(1-x)]}=\lim_{x\to 0}\frac{-\frac{1}{2}x^2-\frac{1}{2}x^2}{x^2[x+\ln(1-x)]}=\lim_{x\to 0}\frac{1}{x+\ln(1-x)}=\infty$$

Because $\lim\limits_{x\to 0}\frac{\cos x-1}{e^{\frac{x^2}{2}}-1}=-1\neq 1$, this solution is obtained according to Theorem 7, which is incorrect. Using Taylor expansion near x = 0, the correct result can be obtained.

Solution 2:

$$\lim_{x\to0}\frac{\cos x-e^{\frac{x^2}{2}}}{x^2[x+\ln(1-x)]}=\lim_{x\to0}\frac{\left[1-\frac{x^2}{2}+\frac{x^4}{4!}+o(x^4)\right]-\left[1-\frac{x^2}{2}+\frac{x^4}{8}+o(x^4)\right]}{x^2[x+\ln(1-x)]}=\lim_{x\to0}\frac{-x^4/12}{x^2[x+\ln(1-x)]}$$

$$=-\frac{1}{12}\lim_{x\to0}\frac{x^2}{x+\ln(1-x)}=-\frac{1}{6}\lim_{x\to0}\frac{x}{1+\frac{-1}{1-x}}=\frac{1}{6}$$

So the right solution is obtained by Using Taylor expansion, while Theorem 7 does not yield the correct result in Example 1. However, Theorem 7 is correct in some other calculations, such as Examples 2 and 3. This example is a counterexample to the theorem. Therefore, Theorem 7 has its imperfections, and other examples in literature happen to avoid the necessary conditions. To discover the problem with Theorem 7, consider the proof of the theorem in literature [3]. Let $\alpha\sim\alpha'$, $\beta\sim\beta'$, if $\lim\gamma=0$ and $\lim\frac{\alpha'}{\beta'}=k\neq1$, then the proof of the theorem in is as follows:

$$\lim\frac{\alpha-\beta}{\gamma}=\lim\frac{\beta\left(\frac{\alpha}{\beta}-1\right)}{\beta'\left(\frac{\alpha'}{\beta'}-1\right)}\cdot\frac{\alpha'-\beta'}{\gamma}=\lim\frac{\beta}{\beta'}\cdot\lim\frac{\frac{\alpha}{\beta}-1}{\frac{\alpha'}{\beta'}-1}\cdot\lim\frac{\alpha'-\beta'}{\gamma}=\lim\frac{\alpha'-\beta'}{\gamma}\#(1)$$

However, the second equation above does not necessarily hold. The condition for the transition from $\lim\frac{\beta\left(\frac{\alpha}{\beta}-1\right)}{\beta'\left(\frac{\alpha'}{\beta'}-1\right)}\cdot\frac{\alpha'-\beta'}{\gamma}$ to $\lim\frac{\beta}{\beta'}\cdot\lim\frac{\frac{\alpha}{\beta}-1}{\frac{\alpha'}{\beta'}-1}\cdot\lim\frac{\alpha'-\beta'}{\gamma}$ is that all three limits $\lim\frac{\beta}{\beta'}$, $\lim\frac{\frac{\alpha}{\beta}-1}{\frac{\alpha'}{\beta'}-1}$, $\lim\frac{\alpha'-\beta'}{\gamma}$ exist, which is not yet guaranteed by Theorem 7. Returning to example 2, the solution could be

$$\lim_{x\to0}\frac{\cos x-e^{\frac{x^2}{2}}}{x^2[x+\ln(1-x)]}=\lim_{x\to0}\frac{\cos x-1-\left(e^{\frac{x^2}{2}}-1\right)}{x^2[x+\ln(1-x)]}=\lim_{x\to0}\frac{\left(e^{\frac{x^2}{2}}-1\right)\left[\frac{\cos x-1}{e^{\frac{x^2}{2}}-1}-1\right]}{\frac{1}{2}x^2(-1-1)}\cdot\frac{-\frac{1}{2}x^2-\frac{1}{2}x^2}{x^2[x+\ln(1-x)]}$$

The second equation in the above does not hold because the limit on the left exists, while the limit on the right does not. This is the reason for the problem of Theorem 7. Therefore, by analogy with the theorem, Theorem 7 should be enhanced as follows [9].

Theorem: If $\alpha\sim\alpha'$, $\beta\sim\beta'$, when $\lim\frac{\beta}{\alpha}=k\neq1$ and $\lim\frac{\alpha'-\beta'}{\gamma}$ exists, then $\alpha-\beta\sim\alpha'-\beta'$; when $\lim\frac{\beta}{\alpha}=k\neq-1$ and $\lim\frac{\alpha'+\beta'}{\gamma}$ exists, then $\alpha+\beta\sim\alpha'+\beta'$.

The proof of the theorem is as in equation (1). Applying this theorem can fully prove the correctness of equivalent infinitesimal substitution in the calculation of addition and subtraction limits in some literature.

Example 3 : Calculate $\lim_{x\to0}\frac{\sin 3x-\tan 2x}{\sqrt{1+x}-1}$

Solution:

$$\lim_{x\to0}\frac{\sin 3x-\tan 2x}{\sqrt{1+x}-1}=\lim_{x\to0}\frac{3x-2x}{\frac{1}{2}x}=2$$

This calculation process satisfies all the conditions of the theorem: as $x \to 0$, $\sin 3x \sim 3x, \tan 2x \sim 2x$, and $\lim_{x\to 0}\frac{\sin 3x}{\tan 2x} \neq 1$, and $\lim_{x\to 0}\frac{3x-2x}{\sqrt{1+x}-1} = \lim_{x\to 0}\frac{3x-2x}{\frac{1}{2}x}$ exists. According to this theorem, the equivalent infinitesimal substitution can work.

Example 4: Calculate $\lim_{x\to 0}\frac{e^x-e^{\sin x}}{x-\sin x}$.

Solution:

$$\lim_{x\to 0}\frac{e^x-e^{\sin x}}{x-\sin x} = \lim_{x\to 0}\frac{(e^x-1)-(e^{\sin x}-1)}{x-\sin x} = \lim_{x\to 0}\frac{x-\sin x}{x-\sin x} = 1$$

In this example, as $x \to 0$, $e^x-1 \sim x, e^{\sin x}-1 \sim \sin x$, $\lim_{x\to 0}\frac{e^x-1}{e^{\sin x}-1} = \frac{x}{\sin x}$, and $\lim_{x\to 0}\frac{x-\sin x}{x-\sin x}$ exists. According to this theorem, equivalent infinitesimal substitution can be used here. The incorrect solution in Example 3 is precisely because it doesn't satisfy that $\lim\frac{\alpha'+\beta'}{\gamma}$ exists. Examples 1, 3, and 4 satisfy this condition [2].

3 Conclusion

Indeterminate limit calculations are problems encountered when solving limits, and it is important to pay attention to the conditions, under which various methods are used to calculate indeterminate limits. The article affirms the indispensable nature of infinitesimal substitution in the mathematical toolkit, offering a fresh perspective on the computation of indeterminate limits and contributing to the ongoing discourse on mathematical rigor and innovation. The conditions provided by theorems must be sufficient, which is a guarantee that an infinitesimal substitution when solving indeterminate limits will yield correct results. The exploration of indeterminate limit calculation theorems must be rigorous to ensure the correctness of the theorems. This paper corrects and provides a complete theorem for the sufficiency of using equivalent infinitesimal substitution in addition and subtraction in indeterminate limit calculations, providing a theoretical guarantee for the use of equivalent infinitesimal substitution in addition and subtraction. By using the theorem presented in this paper, a re-examination of indeterminate limits in the literature has yielded correct results.

Acknowledgments

The project is supported by the Sichuan Association for Non-Government Education with the project number MBXH23YB85 for the year 2023.

Conflicts of interest

The author declares no conflicts of interest regarding the publication of this paper.

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