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Investigation of ruled surfaces via Frenet frame in 2-dimensional light cone

Jiangxiang GAO*

Department of Mathematics, Kunsan National University, Kunsan 54150, Republic of Korea
*Corresponding author
E-mail address: jiangxiang@kunsan.ac.kr

Abstract: This paper focuses on the developability, minimality and curvature characteristics of ruled surfaces in 2-dimensional light cone Q^2. First, we review the basic curve theory of Q^2. And then, we give the equation of ruled surfaces in Q^2 via Fernet frame and calculate the Gaussian and mean curvature of this ruled surface. Finally, we analyze the condition for developable and minimal ruled surfaces in Q^2.

Keywords: Frenet frame; light cone; ruled surface

1 Introduction

Let $\boldsymbol{E_1^3}$ be the 3-dimensional pseudo-Euclidean space with the metric

$\overline{G}(x,y) = \langle x\,, y\rangle = x_1y_1 + x_2y_2 - x_3y_3,$

where $\boldsymbol{x} = (\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3})$, $\boldsymbol{y} = (\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{y_3}) \in \boldsymbol{E_1^3}$. Let $\mathbf{c} \in \boldsymbol{E_1^3}$, then the pseudo-Riemannian lightlike cone are defined as

$Q_1^2(c) = \{x \in E_1^3 \,|\, \overline{G}(x-c, x-c) = 0\},$

when $\boldsymbol{c} = \mathbf{0}$, we denote $\boldsymbol{Q_1^2}(\mathbf{0}) = \boldsymbol{Q^2}$and call it the 2-dimensional light cone.

Ruled surface is a well known and basic surface in 3-dimensional Euclidean space $\boldsymbol{R^3}$. It is defined as a surface has at least one straight line pass through every point on this surface, the parametric representation of a ruled surface is $\mathbf{S}(\mathbf{u},\mathbf{v}) = \mathbf{b}(\mathbf{u}) + \mathbf{v}\mathbf{d}(\mathbf{u})$, where $\mathbf{b}(\mathbf{u})$ is the base curve and $\mathbf{d}(\mathbf{u})$is the director curve.

There are many researches for ruled surfaces on Euclidean and Minkovski space, while the study about ruled surfaces on light cone are few. In [2] Ali has studied one type of ruled surfaces in light cone, based on previous works, we consider more general ruled surfaces in $\boldsymbol{Q^2}$.

This paper is organized as follows: At first, some basic geometric knowledge is shown in section 2, including the Frenet frame of the curves in $\boldsymbol{Q^2}$ and Gaussian and mean curvatures of ruled surfaces. Then in section 3, we get the condition for developable and minimal ruled surface in $\boldsymbol{Q^2}$ and give some special cases about this ruled surface.

2 Preliminaries

The metric in $\boldsymbol{E_1^3}$ is defined as

$\langle , \rangle = {dx_1}^2 + {dx_2}^2 - {dx_3}^2,$

where $(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3})$ is the coordinate of $\boldsymbol{E_1^3}$. For any vector $\mathbf{v} \in \boldsymbol{E_1^3}$, it is called

$(\boldsymbol{i})$. Null (Lightlike), if $\langle \boldsymbol{v}, \boldsymbol{v}\rangle$=0 and $\boldsymbol{v} \neq \mathbf{0}$.

(ii). Spacelike, if $\langle v, v\rangle > 0$ or $v = \mathbf{0}$.

(iii). Timelike, if $\langle v, v\rangle < 0$.

Specially, a curve $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\boldsymbol{\vartheta})$ is called null, spacelike or timelike if there exists a $\boldsymbol{\varphi}'(\boldsymbol{\vartheta})$ is null, space or timelike.

Given a spacelike regular curve $= \boldsymbol{\gamma}(\mathbf{s}) \subset \boldsymbol{Q}^2 \subset \boldsymbol{E}_1^3$, let $\{\boldsymbol{\gamma}(\mathbf{s}), \mathbf{T}(\mathbf{s}), \mathbf{N}(\mathbf{s})\}$ denote the Frenet frame of $\boldsymbol{\gamma}(\mathbf{s})$, then $\{\boldsymbol{\gamma}(\mathbf{s}), \mathbf{T}(\mathbf{s}), \mathbf{N}(\mathbf{s})\}$ satisfying [1]:

$$\begin{cases} \gamma^{'}(s) = T(s) \\ T^{'}(s) = k(s)\gamma(s) - N(s) \\ N^{'}(s) = -k(s)T(s) \end{cases}$$

where $\boldsymbol{k}(\boldsymbol{s})$ is the cone curvature function and

$\langle\gamma(s), \gamma(s)\rangle = \langle N(s), N(s)\rangle = \langle\gamma(s), T(s)\rangle = \langle N(s), T(s)\rangle = 0,\ \langle T(s), T(s)\rangle = \langle N(s), \gamma(s)\rangle = 1.$

From the definition of ruled surfaces in Enclidean space, a ruled surface in $\boldsymbol{Q}^2$ can be represented as

$$\Sigma(s,u) = \gamma(s) + ul(s),$$

where $\boldsymbol{\gamma}(\mathbf{s})$ is the base curve and $\mathbf{l}(\mathbf{s})$ is the director curve. The unit normal vector field $\mathbf{N}$ on $\boldsymbol{\Sigma}$ are defined as [3]

$$N_\Sigma = \frac{\Sigma_s \times \Sigma_u}{\sqrt{\langle \Sigma_s \times \Sigma_u, \Sigma_s \times \Sigma_u\rangle}},\quad \Sigma_s = \frac{\partial \Sigma}{\partial s},\quad \Sigma_u = \frac{\partial \Sigma}{\partial u}.$$

Hence the first and second fundamental forms of Σ are as follow:

$$I: E = \langle \Sigma_s, \Sigma_s\rangle, F = \langle \Sigma_s, \Sigma_u\rangle, G = \langle \Sigma_u, \Sigma_u\rangle,$$

$$II: e = \langle \Sigma_{ss}, N_\Sigma\rangle, f = \langle \Sigma_{su}, N_\Sigma\rangle, g = \langle \Sigma_{uu}, N_\Sigma\rangle.$$

By using above forms, the Gaussian curvature and mean curvature of $\boldsymbol{\Sigma}$ are calculated as follow:

$$K_\Sigma = \frac{eg-f^2}{EG-F^2},\quad H_\Sigma = \frac{1}{2}\frac{Eg+Ge-2Ff}{EG-F^2}.$$

Lemma 2.1. A ruled surface $\boldsymbol{\Sigma}$ is

(i) developable if and only if $\boldsymbol{K_\Sigma = 0}$,

(ii) minimal if and only if $\boldsymbol{H_\Sigma = 0}$.

3 Characteristics of ruled surfaces in Q^2

In this section, we get the descriptions of ruled surfaces' minimality and developability in Q^2 by using the Frenet frame.

The parametric equation of a ruled surface in Q^2 is

$$\Omega(s,u) = \gamma(s) + u\beta(s), \tag{3.1}$$

where spacelike curve $\gamma(s)$ is the base curve and $\beta(s)$ is the director curve, by using the Frenet frame $\{\gamma(s), T(s), N(s)\}$, it can be rewritten as

$$\Omega(s,u) = \gamma(s) + u[r_1\gamma(s) + r_2T(s) + r_3N(s)],\qquad u, r_1, r_2, r_3 \in R \tag{3.2}$$

The ruled surface Ω's first fundamental forms and unit normal vector field are as follows:

$$E_\Omega = (1 + ur_1 - ur_3k)^2 - 2u^2r_2^2k,$$

$$F_\Omega = r_2,$$

$$G_\Omega = 2r_1r_3 + r_2^2,$$

$$N_\Omega = \frac{m_1\gamma(s) + m_2T(s) + m_3N(s)}{\sqrt{2m_1m_3 + m_2^2}},$$

where

$$m_1 = r_3 + ur_1r_3 + ur_2^2 - ur_3^2k,$$

$$m_2 = -ur_1r_2 - ur_2r_3k,$$

$$m_3 = ur_2^2k - r_1 - ur_1^2 + ur_1r_3k.$$

And the ruled surface Ω's second fundamental forms are:

$$e_\Omega = \frac{\delta_1 m_3 + \delta_2 m_2 + \delta_3 m_1}{\sqrt{2m_1m_3 + m_2^2}},$$

$$f_\Omega = \frac{r_2km_3 + (r_1 - kr_3)m_2 - r_2m_1}{\sqrt{2m_1m_3 + m_2^2}},$$

$$g_\Omega = 0,$$

where

$$\delta_1 = k + ur_1k - ur_3k^2 + ur_2k',$$

$$\delta_2 = 2ukr_2 - uk'r_3,$$

$$\delta_3 = ukr_3 - ur_1 - 1.$$

Through above calculation, the Gaussian curvature and mean curvature are calculated on the following:

$$K_\Omega = -\frac{[r_2km_3 + (r_1 - kr_3)m_2 - r_2m_1]^2}{(2m_1m_3 + m_2^2)\{[(1 + ur_1 - ukr_3)^2 - 2u^2r_2^2k](2r_1r_3 + r_2^2) - r_2^2\}}, \tag{3.3}$$

$$H_\Omega = \frac{(2r_1r_3 + r_2^2)(\delta_1 m_3 + \delta_2 m_2 + \delta_3 m_1) - 2r_2[r_2km_3 + (r_1 - kr_3)m_2 - r_2m_1]}{2\sqrt{2m_1m_3 + m_2^2}\{[(1 + ur_1 - ukr_3)^2 - 2u^2r_2^2k](2r_1r_3 + r_2^2) - r_2^2\}}. \tag{3.4}$$

Theorem 3.1. The ruled surface in Q^2 represented by (3.1) satisfies the following conditions:

(i) The surface is developable if and only if

$$r_2km_3 + (r_1 - kr_3)m_2 - r_2m_1 = 0.$$

(ii) The surface is minimal if and only if

$$(2r_1r_3 + r_2^2)(\delta_1 m_3 + \delta_2 m_2 + \delta_3 m_1) = 2r_2[r_2km_3 + (r_1 - kr_3)m_2 - r_2m_1].$$

As the extension of theorem 3.1, we can give some special ruled surfaces in Q^2 and characteristics about these ruled surfaces.

Case 1. For $r_1 = 1, r_2 = 0, r_3 = 0$ in (3.2), this ruled surface is called γ -ruled surface, its equation is represented by

$$\Omega^\gamma(s,u) = \gamma(s) + u\gamma(s) = (1 + u)\gamma(s). \tag{3.5}$$

From above equation we can know that the γ ruled surface is only generated by the base curve $\gamma(s)$, this means it is a curve in Q^2.

Case 2. For $r_1 = 0, r_2 = 1, r_3 = 0$ in (3.2), i.e, the direct vector $\beta(s)$ is produced by the tangent vector $T(s)$, this surface is called T-ruled surface and its equation is represented as

$$\Omega^T(s,u) = \gamma(s) + uT(s). \tag{3.6}$$

The Gaussian curvature and mean curvature of T -ruled surface is

$$K_{\Omega^T} = \frac{(k^2u - u)^2}{4u^4k^2}, \quad H_{\Omega^T} = \frac{u^2k'k - uk^2 + u}{-4\sqrt{2}u^3k\sqrt{k}}.$$

Theorem 3.2. The T -ruled surface in Q^2 satisfies the following conditions:

(i) The surface is developable if and only if

$$k^2 = 1.$$

(ii) The surface is minimal if and only if

$$k(s) = \sqrt{Ce^{\frac{2s}{u}} + 1}.$$

for constant C.

Proof

(ii) From proposition 2.1, the surface is minimal if and only if

$uk'k - k^2 + 1 = 0,$

by transformation, we can get the following equation

$$\frac{dk}{ds} = \frac{k^2 - 1}{uk},$$

this equation implies

$$k(s) = \sqrt{Ce^{\frac{2s}{u}} + 1}.$$

Case 3. For $r_1 = 0, r_2 = 0, r_3 = 1$ in (3.2), i.e, the direct vector $\beta(s)$ is generated by the normal vector $N(s)$, this surface is called N-ruled surface, its equation is represented as

$$\Omega^N(s, u) = \gamma(s) + uN(s). \tag{3.7}$$

The Gaussian curvature and mean curvature of N-ruled surface is

$$K_{\Omega^N} = 0,\ N_{\Omega^T} = 0,$$

this means the N-ruled surface in Q^2 is both developable and minimal.

4 Conclusion

In this work, we use the Frenet frame to analyze the geometric features of ruled surfaces in 2-dimensional light cone $\boldsymbol{Q^2}$, including developablity, minimality and curvature characteristics, also give some special cases to better understand the results.

According to our work, ruled surfaces in $\boldsymbol{Q^2}$ exhibit broader applicability and higher efficiency compared to prior studies. The developablity and minimality of this ruled surfaces are formulaic and methodological.

Results can be used in differential geometry and physics, mostly applied in general relativity, where light cone is a basic research objects. In the future work, we can investigate ruled surfaces in higher-dimensional light cones, and this research provides a feasible method to further study.

Conflicts of interest

The author declares no conflicts of interest regarding the publication of this paper.

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About the author

Jiangxiang Gao (2000-) male, Han ethnicity, from Wuhan, Hubei Province. Education: Master degree; Research direction: mathematics.